HYDROMECHANICS

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AN ELEMENTARY TREATISE

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CHAPTER I.

Fluids Under Surface Forces.

(1) Hydromechanics is the science which treats of the equilibrium and motion of fluids. It has two divisions—Hydrostatics, which treats of fluids at rest; and Hydrodynamics, which treats of fluids in motion.

Matter exists either in the solid, the liquid, or the gaseous state, although by sufficient changes of temperature and pressure any substance can be made to pass from one of these states to another. Solids have both volume and form and offer marked resistance to any change of either; liquids have definite volume but take the form of whatever holds them; while a gas has neither definite volume nor form and expands until it fills the vessel which contains it.

Liquids and gases have one distinguishing feature, which is absent from solids, and which causes them to be classed together as fluids, *i. e.* a freedom of motion of their particles among one another which causes them to yield to any dividing force and to change their shape under the slightest effort.

A "perfect fluid" is defined as a substance which yields at once to any effort, offering no resistance whatever to change of shape, but no such thing exists in nature. All fluids are more or less viscous and offer some resistance to the movement of their particles among each other, and so, too, all solids are found to be more or less plastic and to flow if acted upon with sufficient force. What distinguishes a plastic solid

from a viscous fluid is that the former requires a certain magnitude of stress to make it yield, while any stress, however small, will make a fluid yield, provided it be applied long enough.

Even gases offer some resistance to change of shape at a finite rate, but, when in equilibrium, gases, all common liquids, and even such viscous substances as honey and tar, behave like perfect fluids. Hence, although the science of hydromechanics is founded upon the consideration of a purely ideal substance, its hydrostatical results are found to be verified in practice, and it is only when fluids in motion are being considered that it becomes necessary to take account of their viscosity.

(2) A "fluid," then, is an ideal substance which has to a perfect degree the distinguishing feature of gases and liquids; *i. e.* it is a substance incapable of resisting change of shape, and, therefore, incapable of experiencing distorting or tangential stress.

It follows directly from this definition that the pressure of a fluid at rest upon any surface must always be entirely a normal pressure.

This may be illustrated by the immersion of a thin plate in water, when it will be found that there is no resistance to the beginning of its motion in any direction in its own plane, though when actual motion takes place there is, of course, a certain amount of frictional resistance varying with the speed.

(3) And, furthermore, it is a necessary consequence of our conception of the nature of fluids that the pressure at any point in a fluid at rest must be equal in all directions. In other words, if an indefinitely small plane area be imagined at any point in a fluid in equi-

librium, not only is the pressure on that area normal to it, but the amount of the pressure is exactly the same no matter how the area is turned about.

For imagine a minute cube of a fluid at rest taken at any point within it, and, neglecting the effect of gravity, consider what would result from the least excess of pressure upon one of its sides. Evidently the fluid would yield and flow through the other five sides. A solid cube only requires for equilibrium that the pressures on opposite sides shall balance, but a fluid cube will not retain its shape unless the same normal pressure is exerted upon each side. Now this reasoning is independent of any assumption as to the angle at which the imaginary cube stands. Therefore, for equilibrium, the pressure must be equal in all directions at each point.

In fact, the principle of sufficient reason justifies the conclusion that the pressure at a point in a fluid acted upon only by surface forces is the same in all directions, since there is no determining cause why it should be greater or less in one direction than in another.

That it is legitimate to neglect the effect of gravity in reasoning as to the equality of the pressures around a point in a fluid will be apparent when it is considered that the weight of any volume of the fluid is proportional to the cube of one of its linear dimensions, while the pressure on its surface is proportional to the square of the same dimension. Thus, as the volume of fluid considered shrinks to a point, its weight vanishes in comparison with the pressure on its surface, and so it is equally true for fluids acted on by gravity as for those acted on only by surface forces, that at any point within them the pressure is equal in all directions.

The equality of pressures in all directions about a point is illustrated by the fact that the pressure per unit area, called the "intensity of pressure," is the same on the bottom of a tank of water as it is on the side of the tank immediately adjoining the bottom. If an appreciable area be taken, the total pressure upon it, as will be hereafter shown, varies with the depth of its center of gravity, but "pressure at a point" means intensity of pressure, or the pressure on an area divided by that area when the area is indefinitely diminished, and it is this pressure which is independent of direction. Thus the pressure on a safety valve in a boiler is the same whether the plane of the valve face be horizontal, vertical or inclined.

(4) In addition to their inability to withstand any tangential or shearing force, all fluids possess another important property. They have absolutely perfect elasticity of volume. That is, every change in the surface pressure acting upon a fluid causes a corresponding change in its volume and density.

But from this it necessarily follows that at every point in a fluid subjected only to surface forces the pressure must be the same. For imagine two contiguous portions of such a fluid to be under unequal pressures; then the part under the greater pressure will be of greater density than the part under the less pressure; but, if so, the former will expand and press against the latter until the forces of elasticity, lessening on the one side and increasing on the other, balance at the bounding surface, and both portions are equally compressed.

(5) There is a marked distinction between gases and liquids in the matter of compressibility, the former

being susceptible of great changes of volume as the surface pressure is altered, while the changes of volume of the latter are almost imperceptible even when the pressure varies from one to the other of attainable extremes. This is the cause of the common but fallacious opinion that liquids are incompressible, and for this reason they are sometimes distinguished from gases as being "incompressible" or "inelastic" fluids, gases being classified as "compressible" or "elastic" fluids. The fact, however, is, as already stated, that every fluid, whether a liquid or a gas, has perfect elasticity of volume; *i. e.* the change in volume per unit volume is proportional to any small change in pressure, whatever the pressure to which the fluid is subjected may be. In other words $\frac{dv}{v} = -kdp$ for all fluids, k being very small and constant, or nearly so, for liquids and being very nearly equal to $\frac{1}{p}$ for gases. Thus for example, the volume of atmospheric air would be reduced approximately $\frac{1}{15}$ by an increase of pressure of one pound per square inch, while the volume of water would only be reduced $\frac{1}{308000}$ part by the same increase of pressure.

(6) It is this perfect elasticity of fluids which causes them to transmit, with undiminished intensity, to every part of their enclosing vessels, any surface pressure which is applied to them at any point. Of course, fluids upon the earth are subjected to the attraction of gravitation, and they may be similarly subjected to other attractive or repulsive forces which act upon the individual particles constituting their mass, and these so-called "bodily forces," proceeding from external

bodies, may, and as will be seen do, produce in fluids pressures which vary from point to point, but omitting them from consideration for the moment, it is evident that there is no other way of producing pressure in a mass of fluid than by applying pressure at some point or points of an enclosing vessel, and it is such surface pressures which the elasticity of fluids causes to be transmitted to all points within them.

Thus the atmospheric pressure, approximately 15 pounds per square inch, is transmitted to every point of a liquid in an open vessel, so that if the liquid had no weight it would exert a pressure of 15 pounds upon every square inch of the vessel enclosing it.

(7) The fact that if a fluid is acted upon only by surface forces the intensity of pressure is uniform all over its surface and at all points throughout its mass can also be derived from a consideration of the normality of fluid pressures. For let Fig. 1 represent a closed vessel full of a fluid upon which gravity does not act, and let the intensity of pressure at A be p. Then, to prove that it is the same at every other point, take a right cylinder of the fluid standing on the surface at A as base, and of unit area cross-section, and suppose it to become solid. It will remain in equilibrium since nothing has occurred to alter the forces acting on it. But the forces acting on its sides, being pressures normal to its surface, have no components along the axis AP.

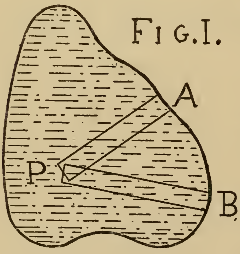


FIG. I.

Hence the pressures at P and at A, each acting at right angles to the unit cross-section of the cylinder, must be equal, or the intensity of pressure at P is the same as at A. But since the pressure is the same in all directions at a point in a fluid, we can now suppose the unit area at P to be turned about so as to be parallel to any other part of the containing vessel, say at B, and then by taking a cylinder of fluid from P to B we see that for equilibrium the pressure on its ends must be equal, and thus the pressure intensity at B is the same as at P, and so on. Hence the pressure is the same at every point in the fluid, and, if we apply a certain pressure at A by means of a piston of one square inch area, we thereby produce that same pressure on every square inch of the surface, so that a piston of 100 square inches area at B could only be held stationary by a force 100 times as great as is applied at A.

This is the principle of the hydraulic press, and by means of it force can be multiplied almost indefinitely, always, of course, at the expense of speed. Thus, in the above example, if 100 pounds is applied to the small piston, the large one will exert a pressure of 10,000 pounds, but the work done by pushing the former in only equals that done by the outward motion of the latter, a movement of one inch of the small only causing one of .01 of an inch on the part of the large piston (neglecting compression of the fluid, which, in the case of a liquid, may be done without sensible error).

If gravity is acting, the weight of the particles of fluid produce, as will be shown, a pressure increasing with the depth, but this pressure is merely superimposed upon that produced by the surface forces, and

in no way affects the principle of the transmission of the surface pressures in all directions with undiminished intensity.

(8) It has already been pointed out that all fluids have perfect elasticity of volume, and that the two classes of fluids, liquids and gases, differ greatly in their compressibility.

A "perfect gas" is defined as one whose compressibility of volume is numerically equal to the intensity of its pressure. In other words, for a perfect gas, $-\frac{dp}{\frac{dv}{v}} = p$, or $\frac{dv}{v} = -\frac{dp}{p}$, whence by integration, $pv =$ constant, or, if v_0 is the volume under pressure p_0, $pv = k = p_0v_0$. Now, while there are no gases in nature which exactly obey this law, which is known as Boyle's, or as Mariottes', law, it is found experimentally that all gases come very near to obeying it, except when near liquefaction, and it may be considered to be practically true.

In the equation $pv = k$, the constant k is only so when the temperature of the gas is constant, otherwise, by the law of Charles, or of Gay-Lussac, $k = k_0 (1 + \alpha t)$, where $\alpha = \frac{1}{273}$ when t is expressed in centigrade degrees. Hence, combining both laws, $\frac{pv}{273 + t} = \frac{p'v'}{273 + t'}$, or, calling T ($= 273 + t$) the absolute temperature, we have for the general equation of the changes in volume, pressure, and temperature of any given mass of any gas, $\frac{pv}{T} =$ constant; and this equation is practically true for all gases except when they are near the point of liquefaction.

(9) In considering the effect of surface pressures upon liquids, unless the utmost refinement of results is desired, they may be considered incompressible, and consequently of unchanging volume and temperature, but, in the case of gases, considerable changes, both of volume and of temperature, may result from a change in the surface pressure. Thus the hydraulic press, while enabling us to multiply force at will, transmits energy, or work, from one point to another without sensible loss, but if it were filled with a gas instead of a liquid, the work done upon the gas, compressing it and raising its temperature, would be an important factor.

Here we will consider only the effect of surface pressures upon gases in the two extreme cases; first, of isothermal, and second, of adiabatic transformation.

In the first case the temperature of the gas is supposed to remain unchanged, or at least to be the same at the end as at the beginning of any change of pressure, and, consequently, we have, under such circumstances, the relation $pv = k = p_0v_0$, or the volume varies inversely as the pressure. If we double the pressure upon any given mass of gas, we half its volume, etc., and this is true provided the pressure is not so great as to have brought the gas near liquefaction, and provided further that the operation is performed in such a way that there is no change in the temperature of the gas.

In the second case we assume that the gas neither gains heat from, nor loses it to, other bodies, and in this case $pv^{\gamma} = k' = p_0v_0^{\gamma}$ (where γ is the ratio of the specific heat at constant pressure to that at constant volume for the gas in question, its value being 1.408 for air and practically the same for all gases), and this

again is true provided the gas is not near liquefaction and provided further that the change in volume takes place either in a non-conducting envelope or so rapidly that the gas has not time to either absorb or give out a perceptible quantity of heat.

In both cases the intensity of pressure is the same at each instant all over the surface and everywhere throughout the mass of the gas, neglecting increase of pressure with depth caused by the weight of the gas itself, which is usually immaterial.

For example, if we have a tube 10 inches long, full of air at 15° C., and at atmospheric pressure (15 pounds per square inch), and if we compress it to a length of 1 inch so slowly that the temperature remains unchanged, the final pressure will be 150 pounds per square inch. If, on the other hand, we compress the same air from 10 inches to 1 inch suddenly, the final pressure will be $15\left(\frac{v_0}{v_1}\right)^\gamma = 15 \times 10^{1.408} = 383.8$ pounds per square inch, and at the same time the temperature will be raised to 458° C., the latter value being given by the formula $\frac{T_1}{T_0} = \frac{p_1 v_1}{p_0 v_0}$, whence $T_1 = (273 + 15)$ $\frac{383.8}{150} = 737$, or $t_1 = 458°$ C.

(10) That the surface of separation of two fluids which do not mix possesses properties peculiar to itself is illustrated by the fact that a needle may be made to float, and certain insects can walk upon the surface of water. This and other so-called capillary actions are caused by molecular forces, which produce a certain surface tension, uniform in all directions, but depending for its amount upon the nature of the fluids in contact. It is this surface tension which makes

fluids sometimes behave as if they were enclosed in an elastic skin, as when, for example, a drop of mercury on a table assumes a nearly spherical form, the effect of gravity, which depends upon the volume, being then small as compared with the tension of the surface. The surface tension, however, differs from that which would result from an elastic skin in not changing in intensity with expansion or contraction of the surface, but remaining always the same per unit length for a particular pair of fluids. Experiment has shown the value of the surface tension, in grains per linear inch of the surface, to be 1.26 for petroleum, 3.23 for water, and 21.53 for mercury, each being in contact with air.

It will be seen that it requires the expenditure of energy to extend the surface of a fluid, and that in contracting its surface a fluid does work, and since, calling the surface tension T, it will require T units of work to increase by unity the length of a strip of fluid surface of unit width, every fluid possesses an intrinsic energy T per unit area of its surface.

Problems I.

(1) If p is the intensity of a uniform fluid pressure, the units being the pound and the inch, what is the total pressure on an area of 2 square feet?

(2) A fluid exerts a uniform pressure on a surface of one square yard. If the total pressure is 5000 pounds, what is the pressure per square inch?

(3) It is desired to test a boiler by an internal pressure of 625 pounds per square inch. What force must be applied to a piston of one inch diameter to produce this?

(4) If the steam pressure in a boiler is 10 atmospheres, taking the atmospheric pressure at 15 pounds per square inch, what weight must be used to load a circular safety valve of $2\frac{1}{2}$ inches diameter to prevent escape of steam?

(5) A closed vessel filled with fluid has a number of pistons fitted in its walls. If one piston, having an area of 10 square inches, is pushed inward with a force of 40 pounds, what change results in the total pressure upon another piston of one square foot area?

(6) If the centers of two circular pistons, respectively of 4 and of 12 inches diameter, are at the same level, and the smaller one is pushed inwards with a force of 100 pounds, how much force must be applied to the larger one to keep it from moving?

(7) If two circular pistons in the walls of a vessel full of fluid have diameters respectively 3 inches and 18 inches, what pressure on the smaller will produce a pressure of 1800 pounds on the larger?

(8) The lever of a hydro-static press is 3 feet long; the distance from its fulcrum to where it acts on the small piston is 8 inches; the diameters of the pistons are respectively one foot and $\frac{1}{2}$ an inch. If a pound is applied to the end of the lever, what weight will the large piston just lift?

(9) What steam pressure in a boiler will just lift its safety valve if the ratio of its arms is 18 to 2 and a weight of 25 pounds is hung on the long arm, the diameter of the valve face being one inch?

(10) If the pressure in the accumulator of a hydraulic forging press is 2 tons per square inch, and the diameter of the press cylinder is 80 inches, what is the power of the press?

(11) The energy of recoil of a 13-inch gun is 280 foot-tons, and it is stopped in 4 feet by a 13-inch hydraulic cylinder whose piston is pushed into the cylinder by an 8-inch rod, the resistance being caused by forcing water from the pressure to the exhaust side through openings designed to produce constant pressure. If the water displaced by the piston rod escapes through spring valves loaded to 800 pounds per square inch, what is the pressure on the recoil side during recoil?

(12) If 100 cubic inches of air at 15 pounds per square inch pressure be raised from a temperature of 20° C. to 50° C., what will be the volume if the pressure is kept constant, and what the pressure if the volume is kept constant?

(13) If the density of the air at the earth's surface is 4 times as great as at a height of 7 miles, and if the temperatures are 22° C. and 0° C., what would the volume of a balloon which held 640 cubic yards of hydrogen gas at the surface be when it had risen 7 miles, provided its envelope offered no resistance to expansion?

(14) If a certain mass of air at 15° C. and 15 pounds per square inch pressure is compressed adiabatically till its temperature is raised to 110° C., what is its pressure and how does its volume compare with its original volume?

(15) What is the pressure of atmospheric air which has been slowly compressed to $\frac{1}{20}$ of its normal volume? and what if it has been very rapidly compressed to the same volume?

(16) If a mass of fluid is falling freely in a vacuum, what shape will it assume, and why?

(17) The 5-meter Whitehead torpedo has an air flask of 11.17 cubic feet capacity, and is charged to a pressure of 1500 pounds per square inch. If atmospheric air at 14.7 pounds pressure weighs .076 pounds per cubic foot, what is the weight of the air in the flask and what would its free volume be?

(18) If rain drops 0″.1 in diameter are each formed by the coalescence of one thousand million smaller drops, how many foot-pounds of energy would be set free if 100 cubic feet of water were formed into drops?

CHAPTER II.

Fluids Under Gravity.

(11) Having seen that the pressure is everywhere the same within and over the entire bounding surface of a fluid acted upon only by surface forces, we will now consider the effect of the force of gravitation upon fluids, that being the only bodily force which acts upon them under ordinary conditions.

In the first place, it is clear that the free surface of a fluid in equilibrium must be horizontal, or at right angles to the direction of the force of gravity. For suppose it is not, but at some point is inclined; then the force acting upon a particle of the fluid at that point can be resolved into rectangular components one of which is parallel to the surface, and since this latter force would be a tangential, or shearing force, the fluid could offer no resistance to it, and so the particle would move; and such unbalanced forces, preventing equilibrium, must exist until the surface everywhere is at right angles to the lines of action of the forces of gravity. The same reasoning, of course, shows that the free surface of a fluid can only be in equilibrium when it is normal at every point to the resultant of all the forces acting on it at that point.

A gas cannot ordinarily have a free surface, but must be kept in a closed vessel to prevent its being diffused, but in any fluid at rest under the action of gravity, whether a liquid or a gas, the pressure is the same at all points in the same horizontal plane. For

suppose A and B are two such points, and imagine a right cylinder of the fluid, of unit cross section, and whose axis is the line joining A and B, to be solidified. Then, since none of the forces acting are in any way changed, it will remain in equilibrium. But the pressures on its curved surface, as well as the forces of gravitation acting upon its particles, are all at right angles to its axis, and so have no components along that axis. Consequently the fluid pressures on its ends must themselves balance and be equal, or the pressures at A and B are equal, and so for all points on the same level within the fluid.

(12) Now consider two close horizontal planes within any fluid at rest, and let p be the pressure at the upper level, $p + dp$ that at the lower level, and dy the distance between them. Imagine a rectangular parallelopiped, of unit cross section, and whose upper and lower bases are in these two levels, to be solidified. It will remain at rest, since the forces acting upon it are unchanged. But the pressures on its opposite vertical sides are normal to those sides and so have no vertical components. Therefore its weight must be balanced by an excess of pressure on its lower over that on its upper base, or, if w is the weight of unit volume of the fluid at the level under consideration, $p + dp = p + wdy$, or $dp = wdy$. If now w be variable, we have $p = \int wdy$, in which w must be expressed in terms of the depth before integration becomes possible, but, if the fluid be homogeneous, so that w is a constant, we have

$$(1) \qquad p = wy + p_0$$

where y is reckoned downward from any assumed level and p_0 is the pressure at that level.

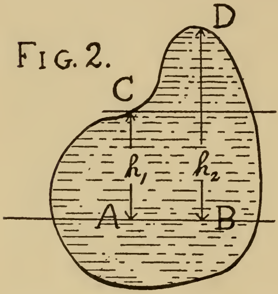
Thus it is seen that in a homogeneous fluid the effect of gravity is to produce an increase of pressure directly proportional to increase of depth. If gravity did not act, the pressure would everywhere be equal to that existing at the surface,—where there was a free surface it would equal the pressure of the atmosphere,—but since gravity always does act, the pressure actually is that at any point of the surface plus wy, where w is the weight of unit volume of the fluid and y is the vertical depth below the selected point on the surface. This is only exactly true when the density is uniform throughout the fluid mass, which, since all fluids diminish in volume and so increase in density as the pressure upon them increases, is never the case. But liquids are so nearly incompressible, and gases are so light, that the change in their densities due to their own weights may be neglected excepting when very great depths are being considered. Practically there are only two such extreme cases—the first when the pressure due to the atmosphere is to be determined, and the second when the effects of the pressure at great depths in the ocean are to be calculated.

(13) The effect of gravitation upon a liquid, then, is to produce at every point within it a pressure depending solely upon its density and the depth of the point below the free surface, and to this must be added any existing surface pressure to give the total pressure at the point. Thus the pressure due to gravity at a point 100 feet below the surface of a lake is 100 x 62.5 pounds per square foot (fresh water weighs 62.5 pounds per cubic foot), or 43.4 pounds per square inch, and to get the actual pressure at that depth the atmospheric pressure on the surface, 15 pounds per square inch,

must be added to this, giving 58.4 pounds per square inch.

In the case of a gas, where there is no free surface, gravitation has exactly the same effect, and the pressure at any point equals the sum of the surface pressure and that due to the depth of the point. Thus in Fig. 2, representing a closed vessel full of a gas, the pressure at the level AB equals the surface pressure at C plus the weight of a column of gas h_1 high, or, what is exactly the same thing, it equals the surface pressure at D plus the weight of a column h_2 high, the surface pressures at C and D differing solely by the weight of a column $h_2 - h_1$ high.

FIG. 2.



So, too, in a liquid, the pressure at any level equals that at any other selected level plus that due to the weight of a column of the liquid whose height is the difference of the two levels, but it is most convenient to take the free surface of a liquid, if, as is usually the case, there be one, as the upper level, and then the pressure at any point is the sum of the pressure on that free surface (usually the atmospheric pressure) and that due to the depth of the point below the free surface, which depth is called the "head" at the point.

(14) It follows that areas of equal pressure in all fluids at rest under gravity are horizontal planes, and that all parts of the free surface of a liquid must be

at the same level however much the continuity of the surface be interrupted by solid bodies. Very large water surfaces, of course, have a spherical contour, being, when at rest, everywhere normal to the direction in which gravity acts. Variations in the atmospheric pressure, however, may produce considerable differences of level at widely separated points.

(15) In the solution of problems it is frequently convenient and useful to apply the principle that areas of equal pressure are level surfaces by regarding even homogeneous fluids as being constituted of layers, each of which produces a surface pressure which is transmitted with undiminished intensity to all points below it. Thus in considering the pressure on any body which is sunk below the surface of a fluid, we can treat it as if just immersed and then merely add to the results obtained on this supposition the effect of the uniform intensity of pressure produced by the layer of fluid which stands above the highest point of the body.

It is of the utmost importance, when numerical results are sought, to express every factor in the units of one particular system and in no others. Thus in calculating the pressure p due to a depth y in a fluid of which unit volume weighs w, by the formula $p = wy + p_0$, if w is in pounds per cubic foot, then y should be expressed in feet and p_0 in pounds per square foot, when p will be found in pounds per square foot. Whereas, if w is the weight of a cubic inch, y should be expressed in inches, and p_0 in pounds per square inch, when the resulting value of p will also be in pounds per square inch.

(16) We will now show that the common surface of two liquids which do not mix must be level. For,

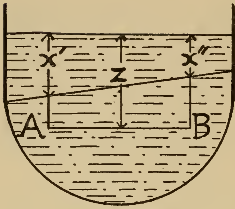


FIG. 3.

referring to Fig. 3, take any two points, A and B, at the same distance Z below the upper surface. Then, since the upper surface has been proved to be necessarily level, the two points A and B are on a level, and therefore the pressure at each must be the same, say p. Now, taking vertical lines at A and B, let x' and x'' be the distances on them from the upper to the common surface, and let w' and w'' be the weights per unit volume of the upper and lower liquids respectively. Then at A we have $p = x'w' + (z - x')w''$, and at B we have $p = x''w' + (z - x'')w''$; whence, subtracting one from the other, $x'(w' - w'') - x''(w' - w'') = 0$. Therefore, since by hypothesis $w' - w''$ is not zero, $x' - x''$ must be, and the common surface, being thus equally distant at every point from the upper surface, must itself be level.

(17) A distinction has thus far been made between liquids and gases by considering that the former alone have free surfaces, but there is one case, that of the earth's atmosphere, where a gas has a free surface, and where, too, the height of that surface above the earth is so great as to make it necessary to take account of the variation in density due to gravitation.

If the density of the air were uniform, the total height of the atmosphere would be given by the formula $p = Hw_0$, or $H = \frac{w_0}{p_0}$, in which H is called the "height of the homogeneous atmosphere," and p_0 and

w_0 are respectively the pressure and the weight of unit volume of air at sea level; and, since p_0 is about 14.7 pounds per square inch and the weight of a cubic foot of air at 0° C. is .0807 pounds, this gives $H = \frac{14.7 \times 144}{.0807} = 26{,}200$ feet (about).

Actually, however, the density of the air diminishes as we ascend, on account of the decreasing pressure, and so a vastly greater height of the atmosphere is requisite to cause the pressure which is found to exist at the earth's surface. Moreover, the density of the air is affected by its temperature, and as this also decreases with increasing height, we can only approximately solve the problems of determining the true pressure at any given height and the true height of the total atmosphere.

(18) Let w_0 be the weight of a cubic foot of air and p_0 the pressure per square foot, at sea level, and let w and p be the corresponding values at the height y. Then $dp = -wdy$; but, assuming the temperature to be constant, $\frac{w}{w_0} = \frac{p}{p_0}$; whence $dp = -\frac{w_0}{p_0} pdy$, or $\frac{dp}{p} = -\frac{w_0}{p_0} dy = -\frac{1}{H} dy$, the integration of which gives us

$$(2) \qquad \log_e p - \log_e p_0 = -\frac{y}{H}, \text{ or } p = p_0 e^{-\frac{y}{H}}$$

From the first of (2), if p_1 and p_2 are the pressures at heights y_1 and y_2, we get by subtraction $y_2 - y_1 = \frac{H}{\mu} (\log_{10} p_1 - \log_{10} p_2)$, where μ is the modulus of the common system of logarithms. But pressures are proportional to barometer readings, and $\frac{H}{\mu} =$ about

60,300 feet = about 10,000 fathoms, so we can write

$$(3) \quad y_2 - y_1 = 10{,}000 \, (\log b_1 - \log b_2) \text{ fathoms,}$$

and we have the simple rule that the difference in height between two places, in fathoms, is 10,000 times the difference of the common logarithms of the simultaneous barometer readings at the two places.

Of course an exact formula for determining heights by the difference of barometer readings requires a number of corrections to be applied, the most important being for difference of temperature between the two places, for the variation in the force of gravity due to height, and for the effect of aqueous vapor in the air on the pressures. Our value of H is based upon the assumption that the atmosphere consists of pure dry air at a uniform temperature of 0° C., but for exact results the value of H corresponding to the actual average conditions between the two places at which observations are taken must be used.

Again, putting (2) in the form $\log_e \frac{p}{p_0} = -\frac{w_0}{p_0} y$, and changing to common logarithms, we have

$$(4) \qquad y = -10{,}000 \log \frac{p}{p_0} \text{ fathoms.}$$

Thus, to find the depth below sea-level at which the pressure is two atmospheres, we have $y = -10{,}000 \log 2 = -3010$ fathoms = about 18,000 feet.

(19) From the formula $p = p_0 e^{-\frac{y}{H}}$ we see that an infinite height is necessary to give p a zero value, which would indicate that there is no limit to the height of our atmosphere. But this is doubtless due to the erroneous assumption of a uniform temperature. In fact an extreme upper limit to the height of the atmosphere is

found to be about $5\frac{1}{2}$ times the radius of the earth, because at that height the centrifugal force due to the earth's rotation would balance the attraction of gravitation and so the particles of air, if there were any there, would fly off into space.

If the laws which govern the fall of temperature with increasing altitude were known, and its effect allowed for, it is probable that the atmosphere would be found to extend at most but a few hundred miles above the earth's surface. It is at least certain that our formula assuming a uniform temperature errs on the side of overestimating the pressure at any given height, and as by that formula the height corresponding to a pressure of only .01 pounds per square foot is about 60 miles, it can be seen how almost infinitely tenuous the air becomes at such heights.

The lowest actual barometric record ever made was in the balloon ascent of Glaisher and Coxwell in 1862, when, at an estimated height of 7 miles the barometer stood at 7 inches and the thermometer at —12° F., indicating a pressure of about $3\frac{1}{2}$ pounds per square inch and a density of the air less than one-fourth that at sea-level.

Problems II.

(1) What is the pressure on an area of one square inch 40 feet below the surface of a lake? What on a square foot?

(2) What depth of fresh water, and what of sea water would cause a pressure equivalent to that of the atmosphere (weights per cubic foot respectively $62\frac{1}{2}$ and 64 pounds and atmospheric pressure 14.7 pounds per square inch)?

(3) What is the intensity of the pressure 1000 fathoms deep in the sea?

(4) Prove that, if two fluids which do not mix meet in a bent tube, the heights of their upper surfaces above their common surface are inversely as their densities.

(5) If a cubic inch of mercury weighs half a pound, what is the pressure on the bottom of a vessel 5 inches deep, full of mercury, and having a bottom of 1 square foot area?

(6) A cubical tank one yard on a side is full of water and has a pipe leading into it which is itself full of water to a height of 10 feet above the surface of the water in the tank. What weight must be put on the top of the tank to keep it closed?

(7) What is the pressure per square inch at the bottom of a vessel which contains water to a depth of 6 inches, and oil 10 inches deep on top of that, if the oil weighs 0.9 as much as the water per unit of volume?

(8) A vessel 8 inches high contains sea water to a depth of 5 inches, olive oil one inch deep on that, and alcohol above that to the top. What is the intensity of pressure at the bottom, the ratios of the respective weights of the three fluids to that of an equal volume of fresh water being 1.027, 0.915 and 0.795?

(9) Equal volumes of oil and alcohol are poured into a circular tube so as to fill half the circle. Show that the common surface rests at a point whose angular distance from the lowest point is $\tan^{-1}\frac{4}{57}$?

(10) If two fluids are put in a circular tube so that each occupies 90°, and the diameter joining the two open surfaces is inclined at 60° to the vertical, find the ratio of the densities of the fluids.

(11) If the average depth of the ocean is 12,000 feet,

and water is reduced one thirtieth in volume for every 10,000 pounds pressure per square inch, how much would the surface rise if gravitation ceased to act?

(12) Given the volume of an air bubble 40′ below the surface of a lake, where the temperature is 4.4° C., what will it be when it rises to the surface and has a temperature of 10° C.?

(13) A circular hollow cone with open base, of weight enough to sink, is suspended with axis vertical and vertex at a given depth below the surface of the water; what is the volume of the air compressed in the cone if its temperature remains unchanged?

(14) What would be the volume of the compressed air in the preceding case were the cone lowered to its immersed position very suddenly?

(15) A cylindrical diving bell of height a is sunk in water till the water rises half way up its axis. Show that the depth of its top is $h - \frac{a}{2}$, h being the height of a water barometer, and show that, if the temperature of the air be raised t° C., the water will recede approximately $\frac{2\,a\,h\,\alpha\,t}{4\,h + a}$.

(16) The barometer reading at the foot of a mountain is 29.75 and at its top is 22.80; what is the difference of level?

(17) How deep must a mine be that the pressure of the atmosphere at its bottom may be $2\frac{1}{2}$ times that at its mouth?

(18) At what depth in water would the air in a bubble equal in density the surrounding water, given that water has 800 times the density of atmospheric air?

(19) If an air bubble has a volume of 3 cubic inches

at a depth of 10′ in water, at what depth will its volume be 2 cubic inches?

(20) A balloon half full of coal gas just floats in the air when the barometer stands at 30; what will happen if the barometer falls to 28? What would happen if the balloon had been full at the first pressure?

(21) A weightless piston fits into a vertical cylinder, of height h and cross section A, closed at its base and full of atmospheric air. The piston being originally at the top of the cylinder water is poured slowly on top of it. How much can be poured in before it will run over?

(22) Equal quantities of fluids, of densities ρ and σ, fill half a small uniform circular tube in a vertical plane. Show that the radius to the common surface makes with the vertical $\tan^{-1} \dfrac{\rho - \sigma}{\rho + \sigma}$.

(23) A circular tube of small uniform bore is half filled with equal volumes of four liquids which do not mix and whose densities are in the ratios 1 : 4 : 8 : 7. Show that the diameter joining the free surfaces makes $\tan^{-1} 2$ with the vertical.

(24) A cylinder is filled with equal volumes of n different liquids which do not mix. If the density of the upper liquid is ρ; that of the next 2ρ; of the next 3ρ; and so on; prove that the mean pressure on the corresponding portions of the curved surface are as $1^2 : 2^2 : 3^2 \ldots : n^2$.

CHAPTER III.

Total Pressures—Centers of Pressure of Plane Areas.

(20) Having learned that the pressure at any point in a fluid is independent of direction, that it is normal to any surface in contact with a fluid, and that its intensity is the sum of the surface pressure and a pressure due to the weight of a column of the fluid of height equal to the depth of the point below the surface, it is next in order to determine the amount of the pressure on any given immersed surface and its line of action.

Let S be any surface, whether plane or curved, and, taking axes of X and Y in any horizontal plane and the Z axis vertical, let p_0 be the pressure at the level XY and let p be the pressure at any point of the immersed surface. Then $p = p_0 + wz$, where w is the weight of unit volume of the fluid, and, if ds be an element of the surface, the whole pressure on the surface is $P = \iint pds$, the integration extending over the whole surface. Substituting the value of p, we have $P = \iint (p_0 + wz)ds = p_0S + w\iint zds$. But $\iint zds = \bar{z}s$, where $\bar{z}$ is the distance of the center of gravity of the surface from the plane of XY. Hence we have

$$(5) \qquad P = S(p_0 + w\bar{z}).$$

In other words the total pressure is the same as if the whole surface were concentrated at its center of gravity.

In the case of liquids, having a free surface, and neglecting the atmospheric pressure, this formula may be stated in words as follows: The whole pressure on any immersed surface equals the weight of a column of the liquid whose cross-section equals the area of the surface and whose height is the depth of the center of gravity of the surface below the surface of the liquid. And this rule applies to all surfaces, whether plain, curved or warped, and however they may be situated with respect to the surface of the fluid.

Thus, for example, the total pressure on the surface of a sphere of radius a just immersed in a liquid is $4\pi a^2 \times wa = 4\pi a^3 w$, the center of gravity of a spherical surface being, of course, at its center. Again, the center of gravity of a zone being midway of its altitude, the total pressure on the upper hemisphere is $2\pi a^2 \times \frac{wa}{2} = \pi a^3 w$, while that on the lower one is $2\pi a^2 \times \frac{3wa}{2} = 3\pi a^3 w$, the sum of these being the total pressure on the sphere as before.

(21) It is evident, too, that in the case of a vessel containing fluid, the whole pressure on the surface is exactly the same in amount as if the vessel were immersed in fluid to the same depth, the only difference being that in the first case the pressure is from in out and in the second case from out in; this, of course, on the supposition that the walls of the vessel have no thickness. For, if the immersed vessel be filled to the level of the outside fluid surface, there must be complete equilibrium, since by supposition the vessel is merely a geometrical locus, as a spherical surface for example, and so the inside pressures must just balance the outside and vice versa.

(22) It is important to note, however, that this total pressure on a curved surface is merely the sum of all the pressures acting on the elements of the surface, which, unless the surface be a plane, have various directions, and there is no physical reality which corresponds to such a sum. It is only the total pressure on a plane surface, or the sum of the components in some fixed direction of the pressures on a curved surface, which has a distinct physical meaning, being the total force with which the surface is urged to move in that direction.

For example, let us determine the sum of the vertical components of the pressures on a sphere just immersed in a fluid. Fig. 4 represents a vertical section of the sphere, whose radius is a, and the equation of this section referred to axes as shown is $x^2 + y^2 = a^2$. If now P be any point on this circle, its depth below the surface is a— y (y being negative for the point P in the figure), and the pressure is the same at all points on the small circle of the sphere whose diameter is PP′. Therefore, if we take an element of the vertical circle, $ds = ad\theta$, the total pressure on a strip of the sphere of that width, and at the level of P, is $2\pi x ds \times (a - y)w = 2\pi awx(a - y)d\theta$ $= 2\pi a^3 w \sin\theta\,(1 + \cos\theta)\,d\,\theta$. But, since every element which goes to make up this total pressure is normal to

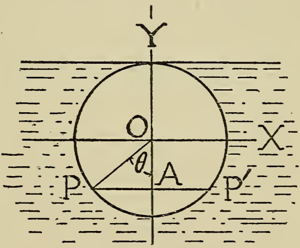


FIG. 4.

the sphere, each of them makes the angle θ with the vertical, and so the vertical component of the total pressure on the strip is $2\pi a^3 w \sin\theta \cos\theta(1+\cos\theta)d\theta$, and the total upward pressure on the sphere is

$$2\pi a^3 w\int_0^\pi \sin\theta\cos\theta(1+\cos\theta)d\theta = 2\pi a^3 w\left[\frac{\sin^2\theta}{2} - \frac{\cos^3\theta}{3}\right]_0^\pi$$

$= \frac{4\pi a^3 w}{3}$. Which, as of course it should be, is the weight of the fluid contents of the sphere.

The total pressure on the sphere has been shown to be $4\pi a^3 w$, or three times as much as the upward pressure just found, but the former is nothing but the sum of the normal pressures acting all over the surface of the sphere, and is incapable of being represented by any single resultant, while the latter is the sum of the forces which urge the immersed sphere upwards, and can be represented by a single resultant force acting, as will be hereafter seen, through the center of the sphere.

(23) It must be remembered that it is the depth of the center of gravity of the surface, not of the solid it may enclose, which determines the total pressure. Thus the whole pressure on the curved surface of a cone of radius of base a, height h, and vertical angle 2α, immersed with axis vertical and vertex at the surface, is $\frac{2\pi ah\sec\alpha}{2} \times \frac{2hw}{3}$, the center of gravity of the surface being at two-thirds the altitude from the vertex.

(24) The next point to consider, having learned how to determine the amount of the pressure on an immersed surface, is at what point does it act. In other words at what point of a surface can a single force

equal to the total pressure on the surface act so as to completely replace the system of forces which are the pressures on the various elements of the surface.

Now in general a system of non-coplanar forces can only be replaced by a single force when the said forces are parallel. It is true that the fluid pressures on any closed surface have a single resultant which acts through the center of gravity of the enclosed volume, but this special case will be considered hereafter. For the moment we will only consider the case of plane areas under fluid pressure, and the problem of determining the line of action of the resultant of the pressures on such surfaces amounts simply to finding where the resultant of a system of parallel forces acts, since all the fluid pressures are normal to the surface and therefore parallel.

(25) The point of action of the resultant of the fluid pressures on a plane area is called the "center of pressure" of the area. If the area is horizontal, its center of pressure is evidently at its center of gravity. Otherwise it must always be below that point, since the fluid pressure increases with the depth.

Suppose $F(xy) = 0$ to be the equation to the curve enclosing the given area, the axis of x being the intersection of the plane of the area with the surface of the liquid, and α being the angle that plane makes with a horizontal plane. Then, if p is the pressure at any point whose coordinates are x, y, $p = wy \sin \alpha$ (neglecting surface pressure); the pressure on an element of the area is pdxdy; the moment of that pressure about the axis of x is pydxdy; and the sum of the moments of all the fluid pressures is $\iint pydxdy =$

$w \sin \alpha \iint y^2 dxdy$. Hence, if $\bar{y}$ be the ordinate of the center of pressure, and P be the total pressure on the area, we have

$$(6) \qquad \bar{y}P = w \sin \alpha \iint y^2 dxdy$$

And similarly, taking moments about the axis of Y, we have

$$(7) \qquad \bar{x}P = w \sin \alpha \iint xydxdy.$$

Usually the total pressure P is most readily found by multiplying the area (A) by wh, where h is the depth of the center of gravity of the area, but when either A or h is not already known, we have

$$(8) \qquad P = w \sin \alpha \iint ydxdy,$$

the integration, of course, extending in all cases over the whole area under consideration.

It will be observed that $\sin \alpha$ cancels out in the determination of $\bar{x}$ and $\bar{y}$, showing that the position of the center of pressure *in the plane of the area* is independent of the angular position of the plane, and this is as it should be, since swinging the plane about its intersection with the liquid surface makes exactly the same proportional change in the pressure upon every element as it does in the total pressure.

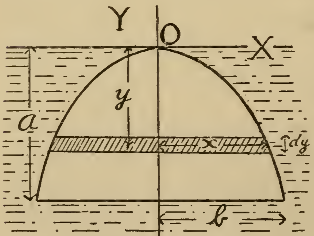


FIG. 5.

(26) It is often unnecessary to resort to double integration to find centers of pressure, since an elementary strip, all parts of which are at an equal depth, and upon which the pressure is consequently uniform, can usually be selected, and the total pressure upon it, and its moment, be at once written down. For example, take the case of a segment of a parabola immersed vertically with vertex at the surface, as shown in Fig. 5, the equation to the curve being $x^2 = \frac{b^2y}{a}$.

If now we take an elementary strip, area 2xdy, the intensity of the pressure all over it will be wy, and so the whole pressure on it will be 2wxydy, and the moment of that pressure about the axis of x will be $2wxy^2dy$. Hence $\bar{y}P = \int_o^a 2wxy^2dy = \frac{2wb}{a^{\frac{1}{2}}}\int_o^a y^{\frac{5}{2}}dy = \frac{4wa^3b}{7}$. But $P = 2w\int_o^a xydy = \frac{2wb}{a^{\frac{1}{2}}}\int_o^a y^{\frac{3}{2}}dy = \frac{4wa^2b}{5}$. Therefore $\bar{y} = \frac{5a}{7}$ and, the curve being symmetrical to the axis of Y, the center of pressure is on that axis and $\bar{x} = o$.

As another example, take a triangle of base b and altitude a, immersed as shown in Fig. 6. Let x be the width of the triangle at the distance y from its vertex. Then the pressure on the elementary strip xdy is

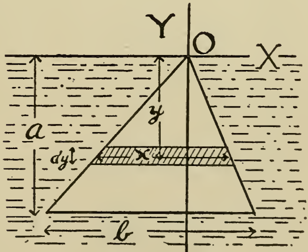


FIG. 6.

wyxdy and its moment about X is wy^2xdy, and the sum of the moments of all the pressure on the triangle is $\int_0^a wy^2xdy = \frac{wb}{a}\int_0^a y^3dy \left(\text{since } \frac{x}{b} = \frac{y}{a} \right) = \frac{wa^3b}{4}$. But $P = \frac{ab}{2} \times \frac{2aw}{3} = \frac{wa^2b}{3}$. Hence $\bar{y} = \frac{wa^3b}{4} \div \frac{wa^2b}{3} = \frac{3a}{4}$.

Sometimes polar coordinates lend themselves better than rectangular to the determination of centers of pressure. Thus to find the center of pressure of a circular disc of radius a vertically immersed with its upper edge tangent to the surface. Taking the origin at the surface, which is almost invariably advisable,

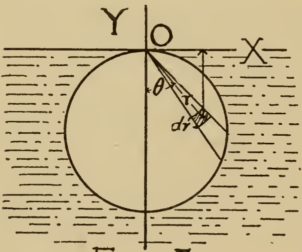


FIG. 7.

the equation to the circle is $r = 2a \cos \theta$. Then taking $rd\theta\, dr$ for the element of area, the intensity of the pressure upon it is $wr \cos \theta$, since $r \cos \theta$ is its depth below the surface; the total pressure upon it is $wr^2 \cos\theta\, dr\, d\theta$; the moment of that pressure about x is $wr^3 \cos^2 \theta\, drd\theta$; and so we have $\bar{y}P =$
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2a\cos\theta} wr^3 \cos^2 \theta\, drd\theta = 4wa^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6 \theta\, d\theta = 4wa^4 \times \frac{5\pi}{16}$$
$= \frac{5\pi a^4 w}{4}$. But we also have $P = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2a\cos\theta} wr^2 \cos \theta\, drd\theta = \frac{8wa^3}{3}$
$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 \theta\, d\theta = \frac{8wa^3}{3} \times \frac{3\pi}{8} = \pi a^3 w$. Therefore $\bar{y} = \frac{5\pi a^4 w}{4\pi a^3 w} = \frac{5a}{4}$. The value of P, the total pressure on the

circle, might, of course, have been written down directly, without the use of the double integral, since it is merely the weight of a column of the fluid whose cross-section equals the area of the circle and whose height is the depth of the center of gravity of the circle.

(27) In the calculations of centers of pressure thus far made the areas have been taken as just immersed, for the reason that this is the simplest case, but, of course, the position of a center of pressure varies with the depth, and it is frequently necessary to determine it when the area is sunk below the surface. This can, however, be easily done by considering that the fluid above the highest point of the area plays the part of an atmosphere, causing a uniform intensity of pressure at every point below it. It is only necessary, then, to combine the resultant of the variable pressures due to the fluid actually in contact with an area with the resultant of the uniform pressure due to the fluid above the area, and this will give the total pressure and the center of pressure of the area in its sunk position.

Thus, for example, if a vertical circle of radius a is immersed with its highest point c below the surface, as in Fig. 8, the uniform intensity of pressure due to the superimposed fluid would be cw, and the resultant would be a pressure πa^2cw acting at the center of gravity of the circle. But the pressure due to being

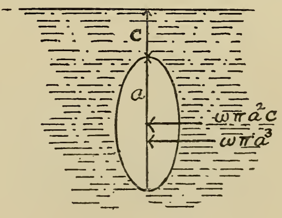


FIG. 8.

just immersed is $\pi a^3 w$ and acts at a distance $\frac{a}{4}$ below the center of the circle. Therefore the total pressure on the circle is $w\pi a^2(a + c)$ and it acts at a distance $\frac{a^2}{4(a + c)}$ below the center, that being, therefore, the position of the center of pressure of a circle placed with its highest point c below the surface of a fluid.

As a rule the atmospheric pressure acts upon both sides of the surfaces whose centers of pressure have to be determined, and so can be left out of account, but in case one side of a surface immersed in liquid has no fluid pressure whatever upon it, the uniform pressure upon the other side due to the atmospheric pressure on the liquid surface must be combined with the pressure of the liquid itself in the same way as has just been explained.

(28) Among practical problems requiring knowledge of centers of pressure is that relating to the stability of brick or masonry embankments against one or both sides of which water stands. For example, we will determine how high water can stand on one side of a brick wall 12 feet high and 3 feet thick, the weight of the wall being 112 pounds per cubic foot. Let h be the height of the water when just about to push the wall over. Then the pressure P on each foot length of the wall is $h \times \frac{hw}{2} = 31.25h^2$; and this pressure acts $\frac{h}{3}$ above the base of the wall, since

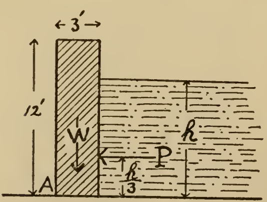


FIG. 9.

the center of pressure of a rectangle just immersed is $\frac{2}{3}$ its altitude below the surface. Now the wall may give way either by sliding horizontally or by turning about its outer lower edge at A, and in practice failure will first occur by the latter method. Therefore, when the wall is just about to upset, the moment about A of the weight of the wall must just equal the moment about the same edge of P, which is the only upsetting force. Hence $\frac{hP}{3} = \frac{3W}{2}$; or $\frac{31.25h^3}{3} = \frac{3}{2} \times 36 \times 112$, or $h = \frac{12}{5}\sqrt[3]{42} = 8.34$ feet.

(29) Another practical application is to determine the forces acting upon the supports of floodgates, dry-dock caissons, and similar structures. Take, for example, an automatic tide gate (Fig. 10) for draining a salt marsh, its action being to open whenever the head of water on the marsh side exceeds that on the ocean side, and to close when this condition is reversed by the rising of the tide. The pressure on the ocean side is $P_1 = wbh_1 \times \frac{h_1}{2} = \frac{wbh_1^2}{2}$, applied $\frac{h_1}{3}$ above the sill (b being the width of the gate), while that on the marsh side is $P_2 = \frac{wbh_2^2}{2}$ applied $\frac{h_2}{3}$ above the sill. The resultant

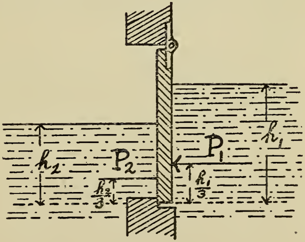


FIG. 10.

pressure, therefore, is $P_1 - P_2 = \frac{wb}{2}(h_1^2 - h_2^2)$, and the height above the sill of its point of application is found, by taking moments about the sill, to be $z = \frac{h_1P_1 - h_2P_2}{3(P_1 - P_2)} = \frac{h_1^3 - h_2^3}{3(h_1^2 - h_2^2)}$. The gate, then, is supported by the hinge and the sill and carries the load $P_1 - P_2$ at the distance z above the sill, so that if R is the thrust on the hinges and a the height of the gate, again taking moments about the sill, we have $Ra = (P_1 - P_2)z$; whence $R = \frac{wb}{6a}(h_1^3 - h_2^3)$.

PROBLEMS III.

(1) A cubical vessel is filled with a homogeneous liquid. Compare the total pressure on its bottom and sides.

(2) What is the total pressure on a solid right cone, radius of base 3′ and height 4′, suspended in water with axis vertical and vertex 10′ below the surface?

(3) A closed hemisphere, full of a fluid, stands with its base vertical; compare the total pressures on its plane and curved surfaces, and show that the sum of the components, perpendicular to the base, of the pressure on the curved surface equals the total pressure on the plane surface.

(4) A triangle is immersed in liquid with its base in the surface; where must a horizontal line be drawn across it so that the total pressure on the two parts shall be equal?

(5) A parallelogram is immersed with one side in the surface; how must a line be drawn from one end of

that side so as to divide the area into two parts, the total pressures on which shall be equal?

(6) A vessel contains mercury with a layer of water 10″ deep standing above it. A rectangular area 8″ high is immersed vertically with its base parallel to and below the common surface of the two liquids. What must its position be so that the pressure of the mercury on the lower shall equal that of the water on the upper part of the rectangle?

(7) A cube full of liquid is suspended by a string tied to one corner. Compare the pressures on its faces.

(8) What is the total pressure in tons on a triangular area of 100 square feet placed with its vertices 4, 8 and 12 feet respectively below the surface of the water?

(9) A vertical water gate 40′ wide has fresh water standing 20′ deep on one side; how deep must salt water stand on the other side that the pressures on the two sides may be equal?

(10) ABCD is a parallelogram whose diagonals AC, BD, intersect at E, and AB is in the surface of a fluid. Show that the pressures on the triangles AEB, BEC, CED are as 1 : 3 : 5.

(11) A hollow sphere is half filled with mercury and then filled up with water; compare the total pressures on the upper and lower hemispheres.

(12) A rectangular sluice gate 12′ square is placed vertically in water, and the pressure on the half cut off by a horizontal line is $\frac{1}{15}$ greater than that on the half bounded by a diagonal. How high is the water above the top of the gate?

(13) A hollow right cone standing base down on a horizontal plane is completely filled with water. Show that the weight of the cone must equal twice that of

the water in order to prevent the cone from rising, and thence deduce the whole normal pressure on the curved surface and the consequent position of the center of gravity of that surface.

(14) A mill race of triangular cross-section (base horizontal, vertex down) is closed by a sluice gate which is supported at the three corners of the triangle. What part of the total pressure on the gate is supported at each corner?

(15) Find the center of pressure of a segment of a parabola immersed with its base in the surface of a fluid.

(16) What is the total pressure on the hinges of an automatic water gate 4′ high by 5′ wide when the sea stands 3′ above the sill on one side and the marsh water 1′ above the sill on the other?

(17) A vertical rectangular masonry dam is 4′ thick and weighs 140 pounds per cubic foot. Determine its height that the water pressure against one side of it may make it fail by sliding, the coefficient of friction being 0.75, and the height at which it will fail by rotation.

(18) A dam of triangular cross section and vertical back is 3′ wide at the base and 15′ high. How high can the water stand before the dam will fail (a) by sliding? (b) by rotation)? the material weighing 140 pounds per cubic foot and the coefficient of friction being 0.75.

(19) The center of a vertical circle of 2′ radius is 10′ below the surface; where is its center of pressure?

(20) Where is the center of pressure of a quadrant of a circle immersed vertically with one edge in the surface?

(21) Find the center of pressure of a rectangle vertically immersed, with one angle in the surface and one diagonal horizontal.

(22) A circular sluice gate is to be hung on a horizontal axis so placed that the gate shall automatically close when water stands at a given height above its center. How far below the center of the gate must the axis be put?

(23) Prove that the pressures on the curved surface of a cylinder full of fluid held with its axis making an angle β with the horizontal have a single resultant which makes the angle β with the vertical.

(24) A hollow cone without weight is filled with mercury and then inverted so as to stand with its open base on a smooth horizontal table. What is the maximum height of the cone which will cause it to remain full?

(25) A heavy conical cup stands vertex up on a smooth level surface and water is gradually poured into it through a hole in its top. If the weight of the cup is $\frac{5}{8}$ that of the water which would fill it, how high will the water be when the cup is on the point of rising?

(26) A cone full of water is suspended by a cord attached to a point on the rim of its base. Prove that the total pressure on its curved surface and that on its base are in the ratio $1 + 11 \sin^2 \alpha : \sin^3 \alpha$, where α is the semivertical angle of the cone.

CHAPTER IV.

Resultant Pressure on Curved Surfaces—Density and Specific Gravity.

(30) We have seen how to determine the sum of all the fluid pressures on any immersed surface, whether plane or curved, and also, in the case of plane surfaces, how to determine the line of action of the single force which is the resultant of those pressures.

In the case of curved surfaces, as already pointed out, the sum of the pressures bears no necessary relation to their resultant, and in order to determine the latter it is usually necessary to first find its components in the vertical and in two horizontal directions.

Take any surface whatever immersed in a liquid, and project it upon the level free surface of the liquid. Imagine now that the liquid contained within the vertical lines whose intersections with the liquid surface form the bounding curve of the projection is solidified. Then the only force which is opposed to the weight of the solidified mass is the vertical resultant of the fluid pressures upon the immersed surface which now forms the base of the solid. Therefore the resultant vertical pressure on any immersed surface equals the weight of the fluid which stands upon the surface, and acts through the center of gravity of that fluid.

Next project the immersed surface upon any vertical plane and imagine the liquid within the bounding, horizontal, projecting lines to be solidified. Evidently, for equilibrium, the horizontal resultant, in a direction

perpendicular to the assumed plane of projection, of the pressures on the immersed surface must be equal and opposite to the resultant of the pressures on the projection of that surface. Therefore the resultant in a given horizontal direction of the pressures on any immersed surface equals the whole pressure upon the projection of the surface on a plane perpendicular to that direction, and acts at the center of pressure of that projection.

Hence, in general, to determine the resultant of all the fluid pressures on a curved surface, we must find the vertical pressure on the surface, and the horizontal pressure on it in two directions at right angles to each other; and then the resultant of these three forces is the whole resultant fluid pressure on the surface.

This resultant will usually be a dyname, or the combination of a couple and a force which is not parallel to the plane of the couple, since, as a general rule, a system of forces, such as the fluid pressures on a curved or warped surface, which are neither coplanar nor parallel, cannot be balanced by any single force.

(31) There is one case, however, in which the fluid pressures always have a single force as their resultant, and that is when the immersed surface is a completely closed one, so that it forms the boundary of a geometrical solid. For, if such a surface be both surrounded and filled with a fluid, it will be in equilibrium, being itself without weight, and if then its contents be solidified, without change of weight or volume, it will remain in equilibrium, and, consequently, the resultant of all the fluid pressures upon it must be a force equal and opposite to the weight of its contents. In fact, the projection of a closed surface upon any plane,

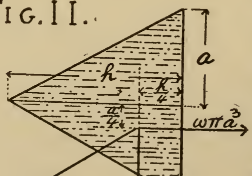
when algebraically considered, is zero, which shows that there is no resultant horizontal pressure on such a surface, and the weight of the fluid which stands upon a closed surface is really the difference of two columns, one representing the upward and the other the downward pressure, their difference, which is the fluid contents of the surface, being the resultant vertical pressure.

(32) The resultant, then, of the pressures on any completely surface immersed in a fluid is a single force, equal to the weight of the fluid which the surface will contain, and acting vertically upwards through the center of gravity of the volume enclosed by the surface.

When, therefore, any curved surface can be completely closed by the addition to it of plane surfaces, we can readily determine the resultant of fluid pressures on the curved part, for the foregoing principle gives at once the resultant of the pressures over the whole closed surface, and since the pressures on each plane face have a single force as a resultant, we have only to combine the latter, reversed in direction, with the former and the result will be the pressure on the curved part of the surface.

Thus, for example, take a right cone full of a fluid and held with its axis horizontal, and let us determine the resultant of the pressures on its curved surface. The resultant of the pressures over the entire closed surface is the weight of the fluid, $\frac{\pi a^2 h}{3}w$, acting at

FIG. 11.



its center of gravity, $\frac{h}{4}$ from the base. The pressures on the base have for resultant the force $\pi a^2 w$ acting at the center of pressure of the base which is $\frac{a}{4}$ below its center. Consequently the pressures on the curved surface of the cone must have a single force as their resultant, that force being the resultant of $\frac{\pi a^2 h}{3} w$ acting as shown in Fig. 11 and $\pi a^2 w$ reversed in direction.

(33) It is rarely necessary, for any practical purpose, to determine the resultant pressure upon curved surfaces excepting in the case of bodies floating wholly or partly immersed in a fluid. But the importance of that case justifies a repetition and enlargement of the argument from which it was shown that the resultant pressure upon a closed surface is equal and opposite to the weight of the fluid it would contain.

If we imagine any portion of a fluid at rest to be solidified, without change of volume or weight, it will remain in equilibrium, and consequently the resultant of all the fluid pressures over its entire surface must be a force equal and opposite to the weight of the solidified body. Now if we put another body, of different weight, but of the same form, in place of the solidified fluid, evidently the pressure on its surface will remain unchanged and so there will now be two forces acting, the weight of the new body acting at its center of gravity and the resultant of the surface pressures, which is equal to the weight of the displaced fluid and acts at its center of gravity. These two forces, one acting upwards and the other downwards, can only

balance when they are equal and when their lines of action coincide. Exactly the same reasoning applies to the case of a body only partially immersed in a fluid; there can only be equilibrium when the resultant of the fluid pressures is equal and opposite to the weight of the body, and, at the same time, that resultant must be equal and opposite to the weight of the fluid which the body displaces.

(34) The resultant of the fluid pressures on a floating body is called the buoyant effort of the fluid, and its point of application, which we have just seen is the center of gravity of the displaced fluid, is called the center of buoyancy of the body.

If, then, a body is either wholly or partially immersed in a fluid, its apparent weight is reduced by the weight of the fluid it displaces, and if it floats in equilibrium, its weight must be exactly equal to the weight of the fluid it displaces, and its center of gravity and center of buoyancy must lie in the same vertical line.

If the fluid is homogeneous, the center of buoyancy will coincide with the center of form of the immersed part of the body.

If, on the other hand, the fluid is heterogeneous, it must, when at rest, consist of horizontal strata, of density increasing with the depth, and the displaced fluid will consist of portions of the same strata, so that the buoyant effort will equal the sum of the weights of the different layers displaced, and the center of buoyancy will necessarily be below the center of form of the displaced fluid.

Thus, when a body floats only partially immersed in water, the buoyant effort equals the sum of the weight

of the water displaced and that of the air displaced, and the center of buoyancy is at the center of gravity of the compound body formed by the displaced portions of both fluids.

However, since the weight of air is only about $\frac{1}{800}$ that of water,—a cubic foot of fresh water weighing $62\frac{1}{2}$ pounds while a cubic foot of air weighs only about $1\frac{1}{4}$ oz.,—it is only in very refined observations that it becomes necessary to take displaced air into consideration.

Moreover, while it is true that all fluids under the influence of gravity are necessarily heterogeneous, and consist of horizontal layers of density increasing with the depth, yet the differences in density due to a moderate depth are so small as to be negligible, and practically the portions of fluids, whether gaseous or liquid, in which bodies are immersed can always be considered homogeneous.

(35) It should be clearly understood that the buoyant effort of fluids is entirely due to the increase of pressure with depth caused by gravitation. If a uniform pressure be applied all over any closed surface it has no resultant; it is the varying pressures on immersed bodies which have the vertical resultant we call buoyancy. Thus, neglecting the increased density of the water due to depth, the buoyant effort of the fluid pressures on a body sunk a thousand fathoms below the surface of the sea is exactly the same as if it were just immersed,—the uniform intensity of pressure caused by the superimposed layer of water, when applied over the whole surface of the body has a zero resultant, and so the resultant of all the pressures on the body is always just equal to the weight of the water it displaces.

(36) Let us now apply the principle of buoyancy to the solution of questions relating to the density of bodies. By "density" we mean the *mass* of unit volume. This term is sometimes improperly used to denote the *weight* of unit volume, but we shall call the latter "specific weight." Thus if ρ is the density of any substance, and w its specific weight, $w = \rho g$, where g is the acceleration of gravity.

"Specific gravity" is relative density as compared with a standard substance, usually pure water at 4° C., and consequently it is proportional to weight per unit volume. Thus, if W and M are the weight and mass of any given volume of a substance, and W_1 and M_1 are the weight and mass of the same volume of water, $\frac{W}{W_1} = \frac{Mg}{M_1 g} = \frac{M}{M_1}$, and the specific gravity of the substance is the number resulting from dividing W by W_1.

(37) It will be seen, therefore, that if a homogeneous body floats in a fluid, its density is to the density of the fluid inversely as its whole volume is to the volume of the fluid it displaces. For, since the floating body is in equilibrium, its weight $\rho g V_1$ must equal the buoyant effort of the fluid, $\rho_1 g V_1$; whence $\frac{\rho}{\rho_1} = \frac{V_1}{V}$. Thus if a homogeneous body floats with $\frac{3}{4}$ of its volume immersed in a fluid, its density must be $\frac{3}{4}$ that of the fluid, and if the fluid be water at the standard temperature, the specific gravity of the body is 0.75

Furthermore, since, as we have seen, the weight of any body immersed in a fluid is diminished by its buoyancy, which itself equals the weight of the fluid displaced, the specific gravity of any body must equal

its true weight divided by its loss of weight in water. Thus if a piece of steel weighing 28.33 pounds is immersed in water and then found to weigh 24.71 pounds, the difference of those weights, 3.62 pounds, is evidently the weight of the steel's volume of water, and therefore the specific gravity of the steel is $\frac{28.33}{3.62} = 7.826$.

Again, if a piece of cork weighing one pound in air is placed in water it will be found to require a force of about $3\frac{1}{6}$ pounds to keep it wholly immersed, whence the specific gravity of cork is $\dfrac{1}{1 + 3\frac{1}{6}} = 0.24$, since the buoyancy, or weight of the cork's volume of water, is evidently $3\frac{1}{6}$ pounds more than the one pound weight of the cork itself.

(38) If the specific gravity of a liquid be required, it is only necessary to take a solid of greater density than either the liquid or water, and to weigh it when immersed in each. Then, since its loss of weight in the two cases will be the respective weights of the same volume of the liquid and of water, the loss of weight in the liquid divided by the loss of weight in water will be the specific gravity of the former.

It is a simple matter, too, to find the volume of a solid, however irregular its shape, by determining its loss of weight when immersed in a liquid of known density, such as water, and, if the solid is of less density than the liquid, the same problem can be solved by attaching to the solid a heavy substance which will sink it. Thus let V be the volume and W the weight of a solid, let w be the weight of unit volume, or "specific weight," of the liquid, and let x be the weight, *when immersed in the liquid,* of a piece

of lead heavy enough to sink the solid. Then, if the solid, with the lead attached, weighs W_1 when immersed in the liquid, we have $x + W - wV = W_1$, or $V = \frac{x + W - W_1}{w}$.

(39) The balloon furnishes a good example of a body floating wholly immersed in a fluid, and in unstable equilibrium, rising when the weight of the air which it displaces exceeds its own weight, and falling when that condition is reversed.

We will consider three cases, the first being the purely ideal one, in which the envelope of the balloon is without thickness or weight, perfectly elastic, and offering no resistance to stretching, and no weights are carried.

If, then, W_0 is the weight of the gas, which in this case is constant, and if at any moment the specific weights of the gas and of the surrounding air are respectively w' and w, the corresponding volume of the balloon is $V = \frac{W_0}{w'}$, and the weight of the air it displaces is $wV = W_0 \frac{w}{w'}$. Therefore the lifting force is $F = W_0\left(\frac{w}{w'} - 1\right)$. But, assuming gas and surrounding air to be always at equal temperatures, since the pressures of the two fluids must always balance, their densities will vary exactly in the same proportion as the balloon rises or falls, and so $\frac{w}{w'}$, is a constant. Hence, supposing the specific weight of the gas to be n times that of air, we have for the lifting force the constant value given by the equation—

$$(9) \qquad F = W_0\left(\frac{1}{n} - 1\right)$$

and such a balloon would rise with the constant acceleration $\frac{Fg}{W_0} = g\left(\frac{1}{n} - 1\right)$. Thus, for example, a pound of hydrogen, which will displace 14 pounds of air at the surface of the earth, will always displace 14 pounds of air, however high it rises, provided merely that it can expand freely.

(40) As the second case, we will suppose the envelope to be of fixed internal volume V_0, and that the total weight carried, including the envelope, is W_1 in air. Then if the balloon is partially filled with the weight W_0 of a gas of specific weight w', the volume of air displaced by the gas will be $\frac{W_0}{w'}$, and the lifting force will be given by the equation—

$$(10) \qquad F = W_0\frac{w}{w'} - W_0 - W_1$$

But in this case $\frac{w}{w'}$ is only constant so long as the gas can expand freely in proportion with the decreasing pressure of the surrounding air: the moment the envelope is full of the expanded gas w' becomes constant, while w continues to decrease as the balloon rises.

There will be a constant acceleration, therefore, until the envelope is fully expanded, after which the lifting force will be $V_0w - W_0 - W_1$. Consequently the balloon will cease to rise when $w = \frac{W_0 + W_1}{V_0}$, or, in other words, when the specific weight of the dis-

placed air equals the total weight of the balloon divided by its capacity.

(41) It will be observed that W_1 was stated to be the "weight in air" of the solid parts, and so is really variable, increasing as the balloon ascends into less and less dense air, but the error resulting from assuming W_1 to be constant is so small that it may be neglected. For, if W_1 be 2000 pounds, supposing the average density of the materials to be the same as that of water, this volume would only be about 32 cubic feet, displacing about $2\frac{1}{2}$ pounds of surface air, and so the increase of W_1, due to the balloon's rising to a great height, could hardly exceed the immaterial amount of $1\frac{1}{2}$ pounds.

PROBLEMS IV.

(1) The s. g. of mercury is 13,596, and 27.73 cu. in. of water weigh one pound; how much does a cubic inch of mercury weigh?

(2) The s. g. of lead is 11.4; what volume of lead will weigh a ton?

(3) A piece of marble of s. g. 2.7 weighs 1000 pounds; what is its volume?

(4) A cubic foot of steel weighs 480 pounds, what will it weigh when immersed in water? What force would be required to keep it wholly immersed in mercury?

(5) A piece of lead weighs 47.48 grains in air and 43.33 grains in water. What is its s. g.?

(6) A piece of platinum, weighing 2 pounds, weighs 1.915 when suspended in olive oil, and 1.907 when suspended in water. What are the s. g.'s of the platinum and of the oil?

(7) A block of wood weighs 48 pounds in air; a piece of lead weighs $18\frac{1}{4}$ pounds in water; when fastened together in water they weigh 4 pounds. What is the s. g. of the wood?

(8) The s. g. of cork is 0.24. What weight will 3 cubic feet of cork sustain in sea water?

(9) A vessel full of water weighs $5\frac{1}{4}$ oz.; a piece of platinum weighing $29\frac{1}{4}$ oz. is put in it and it then weighs 33 oz. What is the s. g. of platinum?

(10) The s. g. of pure gold and of copper are 19.3 and 8.62 respectively. What is the s. g. of an alloy of 11 parts gold to one part copper?

(11) Atmospheric air consists of oxygen and nitrogen in the proportions of 21 to 79 by volume and of 23 to 77 by weight. Compare the densities of the three gases.

(12) A piece of gun metal weighs 1057.9 grains in air, and 934.8 grains in water; what proportions of copper and tin does it contain, if their respective s. g.'s are 8.788 and 7.291?

(13) How many gallons of water must be mixed with 10 gallons of milk of s. g. 1.03 to make the s. g. of the mixture 1.01?

(14) A cannon cast from bronze of s. g. 8.55 weighs 546 pounds in air and 462 pounds in water. Show that there must be a flaw in it and find its volume.

(15) How thick must a hollow copper globe of exterior radius a be in order to just float in water, if the s. g. of copper is 9.0?

(16) A block of ice, volume one cubic yard, floats with $\frac{2}{25}$ of its volume above the surface, and a small piece of granite is seen to be imbedded in the ice.

What is the size of the stone if the s. g. of granite is 2.65 and that of ice 0.918?

(17) A diamond ring weighs $69\frac{1}{2}$ grains in air and $64\frac{1}{2}$ grains in water. The s. g. of the material of the ring being $16\frac{1}{2}$ and that of diamond $3\frac{1}{2}$, what is the weight of the latter?

(18) The s. g. of silver being 10.5 and of copper 8.9, in what proportion by weight must they be combined to form a compound which shall weigh one-ninth more in air than in water?

(19) A crown made of gold and silver loses $\frac{1}{14}$ of its weight in water, while equal weights of pure gold and of pure silver lose respectively $\frac{4}{77}$ and $\frac{2}{21}$; what are the proportions by weight and also by volume of the gold and silver in the crown?

(20) A hollow cylinder full of a liquid is held with axis vertical, determine the resultant pressure on the curved surface on one side of a plane through the axis.

(21) If the cylinder is held with axis horizontal, determine the resultant pressure on the lower half of the curved surface?

(22) A right cone full of liquid is held with axis vertical and vertex down. Determine the resultant of the pressures on the curved surface on either side of a vertical plane through the axis.

(23) If the cone is suspended freely from a point in the rim of its base, prove that the resultant pressures on the curved surface and on the base are in the ratio $\frac{\sqrt{273}}{12}$ if the vertical angle of the cone is 60°.

(24) A spherical shell is full of liquid. Determine the resultant pressure on the curved surface cut off by a vertical plane through the center.

(25) If the plane, instead of being vertical, is inclined at a given angle, determine the resultant pressure on each half of the sphere's surface.

(26) A copper sphere, .01 inch thick and 40 feet in diameter, weighs 2200 pounds. It is charged with 175 pounds of hydrogen, which weighs .0056 pounds per cubic foot at the surface atmospheric pressure of 14.75 pounds per square inch, while the air at the same pressure weighs .0807 pounds per cubic foot. With what acceleration will the sphere rise? What is the pressure of the hydrogen at the beginning? At what height will it equal that of the surrounding air? How high will the sphere rise? What will the pressure be inside and outside of it then?

(27) A right cone full of water rests on its side on a level surface. Find the resultant vertical and horizontal pressures on its curved surface.

(28) A solid hemisphere is just immersed in liquid with its base inclined at $\tan^{-1} 2$ to the surface. Find the resultant pressure on its curved surface.

(29) A cone floats with axis horizontal in a liquid of twice its own density. Find the pressure on its base, and prove that, if θ is the inclination to the vertical of the resultant pressure on the curved surface, $\tan \theta = \dfrac{4 \tan \alpha}{\pi}$, where α is the semi-vertical angle of the cone.

CHAPTER V.

Floating Bodies Continued—Stability.

(42) In the two cases considered in the last chapter the envelope of the balloon was supposed to be completely closed, and in the second of those cases the envelope was assumed to be capable of withstanding the increasing excess of internal over external pressure which would follow the ascent of the balloon above the height at which the envelope was completely expanded. In actual practice it is necessary to maintain equilibrium between the internal and external pressures by having the neck, or lower end, of the balloon constantly open. With this arrangement, if the balloon is not completely filled with gas initially, the action at first is precisely the same as in the preceding case, but, as soon as such a height is reached that the expanded gas has filled the envelope, a new action begins, since further ascent causes an escape of gas at the neck. If the balloon would have held P pounds of the gas, but starts up containing only P′ pounds, then the height at which it is just full is that at which the density of the air is $\frac{P'}{P}$ times its density at the earth. We will suppose the balloon to be completely filled with gas at the earth's surface, since otherwise it will only be necessary to first find the height at which it is just full, and then to take that as a new starting point.

Let W_0, then, be the weight of gas which just fills

the balloon, of capacity V_0; let W_1 be the weight in air of all the solid parts; let w and nw be the respective specific weights of air and of the gas at the height Z, and let W be the weight of gas remaining in the balloon at the same height. Then, since the open neck keeps the gas at equal pressure with the surrounding air, n is a constant, the densities of the two fluids changing in the same proportion as the balloon rises or falls. Moreover, $Vnw_0 = W_0$ and $Vnw = W$ (w_0 and nw_0 being the specific weights of air and of the gas at the earth); so that $\frac{W}{W_0} = \frac{w}{w_0}$; which is merely to say that the weight of a fixed volume of the gas is proportional to its density, which is itself proportional to the density of the surrounding air. We have, then, for the lifting force at height Z, $F = V_0 w - W - W_1 = \frac{W}{n} - W - W_1 = W_0 \frac{w}{w_0}\left(\frac{1}{n} - 1\right) - W_1$; or, since $\frac{w}{w_0} = \frac{p}{p_0}$, where p is the pressure at the height Z and p_0 the pressure at the earth's surface,

$$(11) \qquad F = W_0 \frac{p}{p_0}\left(\frac{1}{n} - 1\right) - W_1$$

The balloon will rise until $F = 0$, or $\frac{p}{p_0} = \frac{W_1}{W_0\left(\frac{1}{n} - 1\right)}$, but we have already seen in Chapter II that $\frac{p}{p_0} = E^{-\frac{z}{H}}$; so we have $-\frac{Z}{H} = \log_e \frac{W_1}{W_0\left(\frac{1}{n} - 1\right)}$; whence $Z = H \log_e \frac{W_0\left(\frac{1}{n} - 1\right)}{W_1}$, or using common logarithms,

$$(12) \qquad Z = 10{,}000 \log \frac{W_0}{W_1}\left(\frac{1}{n} - 1\right) \text{ fathoms.}$$

As an application of these formulæ, let us take a balloon of 28,000 cubic feet capacity, filled with hydrogen, which has a specific weight $\frac{1}{14}$ that of air. Then $n = \frac{1}{14}$, and taking w_0 as .08 pounds per cubic foot, $W_0 = \frac{28000 \times .08}{14} = 160$ pounds. Therefore, the lifting force at starting is $W_0\left(\frac{1}{n} - 1\right) = 160 \times 13 = 2080$ pounds; and, if the weight carried was 1000 pounds, the balloon would rise to the height $10{,}000 \log \frac{2080}{1000} =$ 3180 fathoms $=$ 19080 feet.

(43) In practice the ascent is regulated either by throwing over ballast to lighten the load, or by letting gas escape by means of a valve in the top of the balloon to reduce the lifting force. The pressure of the gas in the upper part of the balloon slightly exceeds that of the external air at the same level, since the pressure of gas and air at the neck of the balloon are always equal, and the weight of the column of gas from neck to top of balloon is less than the weight of a parallel column of the outside air. There is also, of course, a continuous loss of gas by diffusion, which, to some extent, affects the results attained in practice with balloons.

(44) Thus far we have chiefly considered wholly immersed bodies, but most practical questions relate to the buoyancy and stability of bodies which float only partially immersed. We will, therefore, now take up the case of a body floating partly immersed in a heavy liquid, and in its consideration we will neglect

altogether the displaced air as of no influence in comparison with the displaced liquid.

The section of a floating body by the plane of the fluid surface is called the "plane of flotation," the line joining the center of gravity of the body and its center of buoyancy is called the "axis of flotation"; and the depth to which the body is immersed is called the "depth of flotation" or the "draft."

As already pointed out, the axis of flotation must be vertical when the floating body is in equilibrium, and it is also evident that, if both the body and the fluid are homogeneous, the center of buoyancy will coincide with the center of gravity of the immersed part of the body.

When the form of a body is known, its depth of flotation in a given fluid will determine its weight, and, similarly, if its weight is known, the depth of its flotation can be found. Thus, if the length and beam at the water line of a battleship are 435 and 76 feet respectively, the mean draft 24 feet, and the coefficient of fineness 0.66, she will displace $435 \times 76 \times 24 \times 0.66$ cubic feet of water, which, at 64 pounds per cubic foot, will weigh 14,500 tons, which, consequently, is the true weight of the ship and all that she contains. Or, reversing the problem, and having given the actual weight of a ship, we can calculate the number of cubic feet of water which she will displace, and then, from the form of her underwater body, determine her draft.

As a simple example, we will determine the depth of flotation of a homogeneous right cone floating with vertex down. Let w be the specific weight of the fluid and w' that of the cone, and let V and h be the

volume and height of the latter, and x its depth of flotation. Then, since the volumes of similar solids are proportional to the cubes of their corresponding linear dimensions, the volume of the displaced fluid is $\frac{x^3V}{h^3}$. Hence the buoyant effort is $\frac{x^3V}{h^3}w$, and this must equal the weight of the cone Vw'. From which we have $\frac{x^3}{h^3} = \frac{w'}{w}$ or $x = h\sqrt[3]{\frac{w'}{w}}$.

(45) A problem of practical importance is to determine the relation between the displacement of a ship and the change in her draft which results from going from salt into fresh water, or vice-versa.

Let D be the displacement in tons, A the water line area in square feet, and x the change of draft in feet. Then, since sea water weighs 64 and fresh water $62\frac{1}{2}$ pounds per cubic foot, the displacement in cubic feet in salt water is 35D and in fresh water it will be $\frac{64}{62.5} \times 35D$. Hence the change in the displaced volume is $35D(\frac{64}{62.5} - 1)$, and this must equal xA; from which we have $21D = 25xA$, or

$$(13) \qquad x = \frac{21\ D}{25\ A}.$$

(46) If the center of gravity of a floating body is not in the same vertical with its center of buoyancy, the weight of the body and the buoyant effort of the fluid form a couple, which, unless some extraneous force is applied to balance it, must continue to rotate the body until it assumes a position in which the two opposing forces act in one and the same straight line. If, then, a floating body in equilibrium be displaced by turning it through a small angle, its position is

said to have been one of "stable" or of "unstable" equilibrium, according as the couple resulting from its angular displacement tends to turn it back to its original position, or still further away from that position. If, however, a small angular displacement does not produce any couple, the centers of gravity and of buoyancy remaining in the same vertical, then the body is said to be in a position of "neutral" equilibrium.

(47) As the practical questions in regard to stability

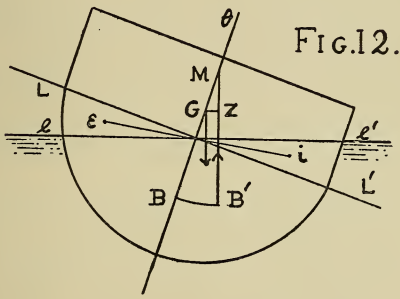


FIG.12.

deal with floating bodies which are symmetrical, when in their normal position of equilibrium, with respect to vertical longitudinal planes through their centers of gravity, only such bodies will be here considered.

Fig. 12 represents a vertical transverse section, through the center of gravity G, of a floating body which has been displaced through the angle θ, ll′ being the new position of the water line, which when the body was in its normal state of equilibrium was LL′. Let V be the displaced volume of liquid, and

suppose the center of buoyancy, which before heeling was at B, is now at B′. Then the weight of the body, acting vertically downwards at G, and the equal buoyant effort of the liquid acting upwards through the new center of buoyancy B′, form a couple, whose arm is GZ, which tends to restore the body to its former position, or to heel it still further, according to whether the point M, the intersection of the new vertical through B′ with the old one through B, is above or below G. In other words, the righting moment is $W \times GZ = W \times GM \sin \theta$, W being the weight of the body and GM being positive when M is above G and negative when M is below G.

The point M is called the transverse metacenter of the body, and so we have the rule that a floating body is in stable, unstable, or neutral equilibrium, according as its metacenter is above, below, or coincident with its center of gravity.

Now the positions of B and G depend entirely upon the form of the body and its distribution of weights, so that the distance BG can be determined in each case. Therefore, if we can determine BM, we have only to compare it with BG to ascertain whether or not the body is in stable equilibrium.

The transfer of the center of gravity of the displaced liquid from B to B′ has been accomplished by removing the wedge of which LOl is a section and inserting the wedge of which L′ol′ is a section, and the volumes of these wedges, which are called respectively the "out" wedge and the "in" wedge, must be equal, since the total displacement remains unchanged. Therefore, if v is the volume of either of

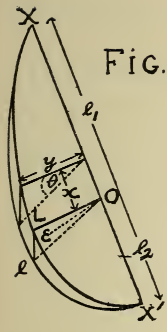


FIG. 13

these wedges, and e and i represent their centers of gravity, we must have $V \times BB' = v \times ei$.

Now let XLX′ (Fig. 13) be half the water-line section, and let y be the half breadth at the distance x, along the longitudinal axis xx′, from the transverse section through G. Then, if θ be small, dv, an element of the volume of the wedge, is $y\theta \times \frac{y}{2} \times dx$, and eo equals $\frac{2}{3}y$; whence we have $v \times eo = \frac{\theta}{3}\int_{-l_2}^{l_1} y^3 dx$, and $V \times BB' = v \times ei = \frac{2\theta}{3}\int_{-l_2}^{l_1} y^3 dx$; or $\frac{BB'}{\theta} = \frac{2}{3V}\int_{-l_2}^{l_1} y^3 dx$. If now θ be indefinitely diminished, $\frac{BB'}{\theta}$ becomes $\frac{ds}{d\theta}$, where ds is an element of the curve BB′ (Fig. 12), and $d\theta$ is the angle between successive normals to that curve; and $BM = B'M = \frac{ds}{d\theta} = \frac{2}{3v}\int_{-l_2}^{l_1} y^3 dx$. Returning now to the one-half water-line plan (Fig. 13), we see that the moment of inertia of that area about its longitudinal axis is $\int_{-l_2}^{l_1}\int_{0}^{y} y^2 dx dy = \frac{1}{3}\int_{-l_2}^{l_1} y^3 dx$, so that we have

$$(14) \qquad BM = \frac{I}{V}$$

where I is the moment of inertia of the whole water-line area about its longitudinal axis.

(48) In order, then, to investigate the stability of any floating body we must calculate the moment of inertia of its water-line area and divide that by its volume of displacement, thus getting the distance from its center of buoyancy to its metacenter; then, subtracting from this result the distance of the center of gravity above the center of buoyancy, we get what is called the "metacentric height," or the distance of the metacenter above the center of gravity, and, if this is positive, the body is in stable equilibrium; if negative, in unstable equilibrium; and if zero, in neutral equilibrium.

(49) If the body be inclined through an increasing angle, the center of buoyancy moves in a curve which is called the curve of buoyancy, and the locus of M, called the metacentric locus, is the evolute of that curve, M being the center of curvature of BB′, and its vertical height above B being always given by the equation $BM = \frac{I}{V}$ where I is the moment of inertia of the momentary water-line area; provided, however, that that area continues to be symmetrical to the vertical longitudinal plane through BM.

With ordinary ship shape forms, however, the point M remains nearly stationary for angles of heel not exceeding about 10°, and for determining the righting moment within that limiting angle, we can practically use the value of BM found for the body's normal position of equilibrium.

(50) As an example, we will now determine the conditions for the stability of a solid cylinder floating

with axis vertical. Let r be the radius, h the height, and h_1 the depth of flotation. The moment of inertia of the water-line area, is $\frac{\pi r^4}{4}$, since it is a circle with a diameter as axis; and $V = \pi r^2 h_1$. Therefore BM $= \frac{r^2}{4h_1}$; and since BG $= \frac{h}{2} - \frac{h_1}{2}$, the metacenter is above the center of gravity as long as $\frac{r^2}{4h_1} > \frac{h - h_1}{2}$, which, accordingly, is the condition for stable equilibrium.

As another example, take a log of square cross section, and of specific gravity S, and let us determine the condition that it may float with one face horizontal. Then if a be one side of the square section, l the length of the log, and h the depth of flotation, $h = Sa$ and $V = ahl = Sa^2l$. Moreover, the water-line area being a rectangle, $I = \frac{a^3l}{12}$. Therefore BM $= \frac{a}{12s}$ Also BG $= \frac{a}{2} - \frac{h}{2} = \frac{a}{2}(1 - S)$. Hence the metacentric height is BM — BG $= \frac{a}{12S} - \frac{a}{2}(1 - S)$. Putting this equal to zero, we have $S^2 - S + \frac{1}{6} = 0$, or $S = \frac{1}{2} \pm \frac{1}{2\sqrt{3}} = .789$ or $.211$. From which we conclude that the specific gravity must either be less than .211 or greater than .789 in order that the metacenter may be above the center of buoyancy, or for stable equilibrium.

Problems V.

(1) A balloon, of capacity 680 cubic yards, starts up with a gross weight of 1150 pounds, including a charge

of 400 pounds of coal gas, the specific weight of which is one-third that of the air. With what acceleration does it rise? How high is it when the gas just fills it? How much higher will it rise without throwing over any ballast? How much gas has then been lost?

(2) A balloon of 8000 cubic feet capacity, and filled with hydrogen, the specific weight of which is $\frac{2}{29}$ that of the air, has to carry a gross weight of 450 pounds (exclusive of gas), including an observer with instruments. How much ballast must be added so that the balloon shall rise with an accelerating force of 10 pounds? How high will the balloon then rise? How high will it rise if all the ballast is thrown over?

(3) A solid right cone floats in a liquid with vertex up; find the position of equilibrium.

(4) A hemispherical vessel of given weight floats in a liquid with one-third its radius immersed. What weight must be put into it to sink it another third of a radius?

(5) A cylindrical pontoon with hemispherical ends, 25′ long and 2′ in diameter, floats half submerged in water. What weight will it support?

(6) A cube floats in a liquid with one corner below the surface and three corners in the surface. Show that the s. g. of the liquid is 6 times that of the cube.

(7) A uniform rod has a small quantity of mercury in one end of it and is found to float half immersed and at any angle with the vertical. Prove that the weight of the mercury equals that of the rod.

(8) How deep will a paraboloid of revolution of height h sink in a fluid whose s. g. is n times its own, the axis being vertical and vertex up?

(9) A cylinder is placed with axis vertical in a

liquid whose density varies as its depth. If the density of the cylinder is the same as that of the liquid at a depth equal to half the height of the cylinder, at what depth will the latter float?

(10) A vessel going from salt into fresh water sinks 2″, but after burning 50 tons of coal rises 1″. What was her displacement?

(11) A 5000-ton ship, drawing 25′, has to discharge 300 tons of water ballast to cross a 24′ bar into a river. She burns 50 tons of coal in going up the river into fresh water. How much is her draft then, and how much ballast will be needed to increase it by 1′?

(12) A cylinder floats in a liquid with its axis inclined to the vertical at an angle $\tan^{-1}\frac{2}{5}$, and with its upper base just above the surface. Show that the radius is $\frac{4}{7}$ the height of the cylinder.

(13) A hollow right cone, with closed base, made of uniform thin material, is found to float wholly immersed in a liquid in any position in which it is placed. Prove that half its vertical angle is $\sin^{-1}\frac{1}{3}$.

(14) A solid right cone floats in water with axis vertical and vertex down. If 2α is its vertical angle, prove that its specific gravity must be greater than $(\cos\alpha)^6$ for stability.

(15) A solid cylinder, one end of which is rounded off with a hemisphere, floats with the spherical surface partly immersed. Determine the maximum height of the cylinder consistent with stability.

(16) A solid paraboloid, height a and radius of base b, floats with axis vertical and vertex down. Show that the height of the metacenter above the center of buoyancy equals half the latus rectum.

(17) A cone whose vertical angle is 60° floats in

water, axis vertical and vertex down. Prove that its metacenter is in the plane of flotation and that its s. g. must be greater than $\frac{27}{64}$ for stability.

(18) A hemisphere floats flat side down. Prove that the distance from its base to its center of gravity and to its metacenter are in the ratio of the densities of the fluid and the solid.

(19) A prism whose right section is an isosceles triangle of given vertical angle floats in water with its vertical angle down. Determine its specific gravity for stable equilibrium.

(20) The right section of a prism is a right angled triangle. If it floats with the right angle down and opposite side horizontal, show that the metacenter is as much above as the center of buoyancy is below the plane of flotation.

(21) Show that a homogeneous sphere will always float in neutral equilibrium, and determine the metacentric height if it is so weighted that its center of gravity is one-fourth the radius from the center.

(22) A cone of density ρ floats with an element vertical in a fluid of density σ, the base being just out of the fluid; show that if 2α is the vertical angle, $\frac{\rho}{\sigma} = (\cos \alpha)^{\frac{2}{3}}$, and that the length of the vertical side immersed is to the length of the axis as $\cos 2\alpha : \cos \alpha$.

(23) A cone floats with axis vertical, vertex down, having $\frac{1}{n}$ of its axis immersed. A weight equal to the weight of the cone is placed upon the base and the cone sinks until just immersed before rising. Prove $n^3 + n^2 + n = 7$.

CHAPTER VI.

Liquids in Uniform and in Steady Motion.

(51) It has already been pointed out that no fluids in nature are "perfect fluids," or incapable of sustaining any tangential stress, but that, on the contrary, all possess to some extent a property, known as viscosity, whereby an internal resistance in the nature of friction opposes itself to their motion and constantly converts into heat some portion of their kinetic energy.

In the study of fluids at rest viscosity may be neglected without any appreciable error, but in the case of fluids in motion its effects are sometimes very great, and must be taken account of. It is necessary, however, to any simple mathematical treatment of the subject of hydrokinetics to neglect viscosity, and we shall therefore confine our mathematical investigations to "perfect fluids," taking account of the viscosity of actual fluids, so far as is possible, by introducing into our results numerical factors determined by experiment. Moreover, we shall, in the main, confine our attention to *liquids* in motion, and for simplicity of treatment we shall consider them as incompressible, which is so nearly the case that no errors of any importance can arise from the assumption.

(52) It has been shown that the pressure intensity at any point in a fluid at rest is equal in all directions, and this is also true in the case of fluids in motion. For, if we take a small cube of a fluid, by D'Alembert's principle the reversed effective forces and the impressed

forces, which act upon the volume of the cube, taken together with the pressures on the faces of the cube, constitute a system in statical equilibrium. Now, if the cube be diminished indefinitely, the effective and the impressed forces, which are proportional to the volume, vanish in comparison with the pressures, which are proportional to the area of a face of the cube, and so finally the cube must be in equilibrium under the pressures upon its faces. But this can only be, in the case of a fluid cube, when the pressure upon each face is the same. Hence, it follows from the essential property of fluids—their inability to resist tangential stress,—that, whether at rest or in motion, at any point within them the pressure intensity is equal in all directions.

(53) A fluid is said to have uniform motion when the velocity of each particle of it is uniform throughout its whole path, and we will first consider, as an example of this simplest case, a fluid rotating about a vertical axis.

If an open vessel containing a fluid be given a uniform rotation about a vertical axis, friction will soon cause the fluid to rotate, like a rigid body, with the same angular velocity, and its free surface will assume a concave shape with its lowest point at its intersection with the axis of rotation.

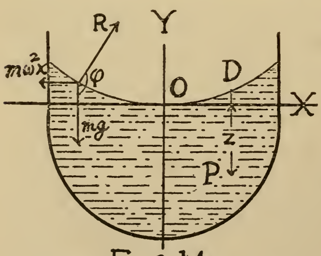


FIG. 14.

Let Fig. 14 be a vertical section of such a vessel, OY being the axis of rotation and ω the uniform angular velocity. Then, taking a particle of the fluid, of mass m, in the free surface, its coordinates being x and y, the impressed forces acting on this particle are mg, its weight, acting vertically down, and $m\omega^2 x$, the centrifugal force, acting at right angles to the axis of rotation as shown; and the surface of the fluid must stand at right angles to the resultant R of these forces. Hence we have $\tan\varphi = \frac{dy}{dx} = \frac{m\omega^2 x}{mg}$; $\int dy = \frac{\omega^2}{g}\int x dx$; $y = \frac{\omega^2 x^2}{2g}$, the equation to a parabola; so that the surface is a paraboloid of revolution.

Moreover, since the motion is uniform, the pressure at any point P in the rotating liquid depends only upon the surface pressure and the depth, so that $p = p_0 + w \times PD$, where p_0 is the surface pressure and w the weight of unit volume of the fluid, whence, if z is the depth below O, we have

$$(15) \qquad p = p_0 + w\left(z + \frac{\omega^2 x^2}{2g}\right)$$

Indeed this last equation can be deduced directly from the conditions of rotation, and thence the equation to the free surface can be obtained in the form $z = -\frac{\omega^2 x^2}{2g}$. For take any point P in the fluid, at the distance r from the axis of rotation and z below the level through o. Then the increase of pressure due to moving a distance dr further out on the same level is $\frac{dp}{dr}.\ dr = \frac{w}{g}\omega^2 r dr$; and that due to an increase of

depth dz is $\frac{dp}{dz}$ dz = wdz. But the pressure is a function of only the two variables r and z. Hence $dp = \frac{dp}{dr} dr + \frac{dp}{dz} dz = \frac{w}{g} \omega^2 r dr + w dz$; or, integrating, $p = wz + \frac{w}{2g} \omega^2 r + C$. But when z = o and r = o, we have $p = p_0$ so that $C = p_0$ and as before, $p = p_0 + w\left(z + \frac{\omega^2 r^2}{2g}\right)$. Moreover, since the pressure must be the same all over the free surface, we get the equation to that surface by putting $p = p_0$, whence $z = -\frac{\omega^2 r^2}{2g}$, showing that it is a paraboloid of revolution with vertex at O.

(54) As another case of uniform motion take the buckets of an overshot water-wheel. Let ABD (Fig. 15) represent a vertical section of a bucket which is moving with uniform angular velocity ω about a horizontal axis perpendicular to the plane of the paper at O. Let P be any point in the section of the water surface AD. Then if m is the mass of a particle of the water at P, the only forces acting on it are mg vertically downwards and $m\omega^2 \times OP$ acting outwards along OP. Representing these forces by PE and PF respectively, their resultant is PG, which cuts the vertical through O at C, and

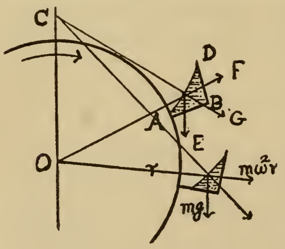


FIG. 15.

from similar triangles we have $\frac{\text{OC}}{\text{OP}} = \frac{\text{mg}}{\text{m}\omega^2 \times \text{OP}}$, or, $\text{OC} = \frac{g}{\omega^2} =$ constant. Therefore, since the water surface must always stand normal to the resultant of all the forces acting on it, each vertical section of the water surface is the arc of a circle whose center is at the fixed distance $\frac{g}{\omega^2}$ above the axis of the wheel, and so the surface in each bucket is a cylindrical one, having for its axis a horizontal parallel to the wheel's axis and at the distance $\frac{g}{\omega^2}$ above it.

(55) The next simplest and by far the most important case to be considered is that of steady motion under the action of gravity.

A fluid is said to be in steady motion when at each point throughout its mass the magnitude and direction of the velocity at that point remains unchanged. In steady motion the lines of motion coincide with the actual paths of the particles of the fluid, which are called "stream lines." If, then, we take a very small closed curve in a fluid in motion, and at each point of its contour drawn a stream line, we have what is called a "stream-tube." At any two points P_1 and P_2 (Fig. 16) in such a stream tube, in a liquid of density ρ, take normal sections, and let their areas be σ and

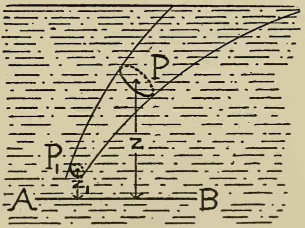


FIG. 16.

σ_1 respectively: let the velocities and pressures at the two points be v, p, and v_1, p_1: let their respective heights above any assumed datum line AB be z and z_1; and suppose that the motion is from P to P_1. Then, since the liquid is incompressible, and there is no motion across the stream lines which form the bounding surface of the tube, a mass $\rho v \sigma$ enters the tube at P in each unit of time and an equal mass $\rho v_1 \sigma_1$ leaves it at P_1: whence we have $v\sigma = v_1\sigma_1$. Now the mass entering at P arrives with a total energy $\rho v \sigma \left(\frac{v^2}{2} + gz\right)$, this being the sum of its kinetic and potential energies, and moreover the work done on the tube at P in each unit of time is $pv\sigma$. Similarly the mass leaving at P_1 carries off the energy $\rho v_1 \sigma_1 \left(\frac{v_1^2}{2} + gz_1\right)$, and the work $p_1 v_1 \sigma_1$ is done by the tube at P_1 while it escapes. But, since the motion is steady, and assuming that there are no losses by friction, the portion of the stream tube being considered neither gains nor loses energy. Hence we must have $pv\sigma + \rho v \sigma \left(\frac{v^2}{2} + gz\right) = p_1 v_1 \sigma_1 + \rho v_1 \sigma_1 \left(\frac{v_1^2}{2} + gz_1\right)$, or, dividing by $g\rho v \sigma = g\rho v_1 \sigma_1$, and writing w, the specific weight of the liquid, for $g\rho_1$.

$$(16) \qquad \frac{p}{w} + \frac{v^2}{2g} + z = \frac{p_1}{w} + \frac{v_1^2}{2g} + z_1$$

In other words, since P and P_1 are any two points along any stream line, $\frac{p}{w} + \frac{v^2}{2g} + z$ is constant for each stream line, though its value may change in passing from one stream line to another.

(56) Evidently $\frac{p}{w}$ is the height of a column of the liquid which will produce by its weight the static pressure p, and this is called the "pressure head." Similarly the quantity $\frac{v^2}{2g}$, being the height in falling from which the velocity v would be acquired, is known as the "velocity head"; and z, the height above an assumed level of zero potential, is called the "potential head."

Thus the mathematical result just deduced, and which is known as Bernoulli's Theorem, amounts to saying that at every point along a stream line in any moving frictionless liquid, the sum of the pressure head, the velocity head, and the potential head, is constant.

(57) It is important to clearly understand the distinction between "hydrostatic" and "hydrodynamic" pressure. If a liquid be at rest the difference between the pressures at any two points within it depends solely upon the difference of the heads at those points, but the moment that motion occurs, the pressure at any point depends not only upon the head but also upon the velocity at that point, and, other things being equal, the greater the velocity the less the pressure and vice versa. Thus, if water stand at the level OO′ in a tank (Fig. 17), and if small open tubes

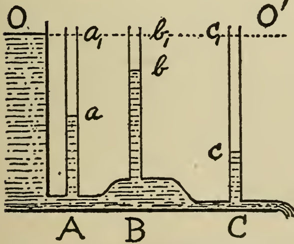


FIG. 17.

(called piezometers) are inserted as shown in the outlet pipe ABC, then, while the water is at rest, it will rise in each tube to the same level OO′ as it has in the tank. But if the outlet be opened so that the water begins to flow, the levels in the tubes will sink to the points a, b and c, showing that the pressures at the three points A, B and C, on the same level, but in different portions of the pipe, are no longer equal, but that the smaller the section of the pipe and the greater, consequently, the velocity of the water, the less is its pressure. The distances aa_1, bb_1 and cc_1 represent the respective velocity heads at A, B and C, and the distances Aa, Bb and Cc are the corresponding pressure heads; the sum of the two heads at each point being constant and, if there are no losses by friction, equal to the original hydrostatic head.

The outflow pipe being full, it is evident that the product of the cross section at any point and the velocity of the liquid at that point must be constant, so that the smaller the section the more rapid must be the motion. But unless the pressure at B were greater than it is at A, the velocity at A would not be retarded to that at B, and, unless the pressure at B were greater than it is at C, the accelerated flow at C would not be produced. If the pressure in a wide section and in a succeeding narrower section were equal, the elements of the liquid would not be accelerated; would not escape fast enough; and so would accumulate at the entrance to the narrower section until a sufficient augmentation of pressure was produced.

(58) We will now apply Bernoulli's theorem to the case of a vessel kept filled to a constant level with

liquid which escapes through a small aperture in its walls. We do this by considering the escaping liquid as a whole to be a stream tube, one normal section of which is at the free surface of the liquid in the vessel and another normal section where the issuing jet attains the atmospheric pressure. Then, if the cross section of the vessel is large as compared with the orifice, we can neglect the velocity at the free surface, and since the pressure there is also the atmospheric pressure, we have at the free surface $\frac{p_0}{w} + z = C$, and in the jet $\frac{v^2}{2g} + \frac{p_0}{w} + z_1 = C$; whence $\frac{v^2}{2g} = z - z_1$ or

$$(17) \quad v^2 = 2gh$$

where v is the velocity of efflux of the liquid and h is the difference of levels of the orifice and the free surface, or the head on the orifice.

This theorem, that the velocity of efflux through an orifice is the same as would be acquired in falling through a height equal to the liquid head on the orifice, was first formulated by Torricelli, and is the foundation upon which the science of hydraulics is largely built. It simply amounts to saying that if there were no losses by friction the energy acquired by a mass of liquid in falling would just suffice to raise it again to the same height from which it fell. When tested experimentally we find that the reversed velocity of efflux almost, but not quite, returns liquid to its original level, and we conclude that the difference is to be accounted for by the viscosity of the liquid.

(59) The importance of the formula $v = \sqrt{2gh}$ will justify its deduction in a more direct manner than by the use of Bernoulli's theorem.

Let an opening of area k be made in the bottom of a vessel containing a liquid of density ρ, and at any time thereafter let K be the cross section of the vessel at the surface of the liquid, h the height of that surface above the orifice, and v the velocity of the escaping liquid. Then, if in the time dt the surface falls dh, in the same interval the liquid mass $K\rho dh$ must escape through the orifice, having the kinetic energy $K\rho dh \frac{v^2}{2}$. But the work done is the descent of the liquid mass $K\rho dh$ from the surface to the orifice, or the distance h. Thus we have $Kg\rho h dh = K\rho dh \frac{v^2}{2}$, or $v^2 = 2gh$.

The only assumption we have made is that all the work done in the vessel appears as kinetic energy in the issuing jet. It is entirely immaterial by what path the liquid moves within the vessel; the whole work done must be equivalent to the transfer of the surface layer to the orifice, and this must equal the energy of the escaping liquid, provided we can neglect losses of energy in overcoming friction and imparting velocity to the liquid remaining in the vessel, and experiments show that we can practically do this if the orifice is small in comparison with the cross section of the vessel.

(60) It will be noted that in the first deduction of the formula $v = \sqrt{2gh}$, v, by hypothesis, is the velocity at a section of the issuing jet throughout which the pressure is the same as at its surface, and such uniformity of pressure can only occur where all the stream lines are parallel and the assumed section a normal one to the stream tube. So also, in the second deduction of $v = \sqrt{2gh}$, v, being merely that

velocity which will account for the kinetic energy of the jet, can only be considered the true velocity of and move in parallel lines. Actually, however, the efflux, if all parts of the jet have the same velocity escaping liquid is made up of a great number of elementary streams converging towards the orifice, and so the motion is not parallel everywhere throughout the area of the orifice, but is more and more oblique to that area as we pass from its center to its boundary. This converging motion of the elementary streams must make the pressure at the orifice somewhat greater in the interior of the jet than at its surface, and consequently the velocity in the interior of the jet must be less than that at its surface.

Mathematical analysis is incompetent to determine either the motions or the actual paths of the issuing particles of liquid, but experiment shows that at a distance beyond the orifice equal to about half its diameter the converging motion ceases and all the particles have equal velocities in practically parallel lines. The reduced section of the jet at this point is called the "vena contracta," and it is there that experiment shows the velocity to be very closely equal to that called for by theory, and given by the formula $v = \sqrt{2gh}$.

(61) The ratio of the cross sectional area of the jet at the vena contracta to that of the orifice is called the "coefficient of contraction," and, as will be seen in the next chapter, varies with the shape and dimensions of the orifice and with the head of the liquid, having an average value of about 0.62 for orifices with sharp inner corners. That it cannot be less than 0.50, in the absence of friction, may be shown as follows:

When the orifice is closed the pressure on it is $(p_0 + wh)a$, where a is its area, h the depth of its center of gravity, p_0 the atmospheric pressure, and w the specific weight of the liquid. When the orifice is opened, the only pressure opposing the exit of the liquid is p_0a, and so there is an unbalanced pressure wha which acts to *expel* the liquid. Now, if c_1 is the coefficient of contraction, the mass of liquid which passes the vena contracta in unit time is $\frac{w}{g}c_1av$, and its momentum is $\frac{w}{g}c_1av^2 = \frac{w}{g}c_1a \times 2gh = 2wc_1ah$; whence, by the fundamental relation $Ft = mv$, we have $wha = 2wc_1ah$, or $c_1 = 0.50$. But really the opening of the orifice produces an unbalanced pressure greater than wha, since the motion of the fluid reduces the pressure on the walls of the vessel near the orifice. Consequently c_1 must really always be greater than 0.50.

Experiment has shown that where a short cylindrical tube, projecting inwards, is attached to the orifice, thus causing the pressure on all parts of the walls of the vessel excepting the orifice itself to remain practically the same with the orifice open as with it closed, the coefficient of contraction actually has its theoretical value 0.50.

PROBLEMS VI.

(1) If the liquid which just fills a hemispherical bowl be made to rotate uniformly about the vertical radius of the bowl, how much will overflow?

(2) A hollow paraboloid of revolution, axis vertical and vertex down, is half filled with liquid. What must

its angular velocity about its axis be in order that the liquid may just rise to the rim?

(3) A hemispherical bowl of radius a containing a given volume of water is set rotating about a vertical diameter. At what angular velocity does the water begin to overflow?

(4) If, in the preceding example, the angular velocity just exceeds $\sqrt{\frac{2g}{a}}$ how much of the bowl is dry?

(5) A narrow horizontal tube AB, with two vertical branches AC and BD, is filled with water to a given height. If this continuous tube be set rotating about a vertical axis through a point O in AB, what will be the difference of level of the water in the two branches?

(6) A closed cylinder just full of liquid rotates uniformly about its vertical axis. What are the total pressures on its top, bottom, and on its curved surface?

(7) If a hollow open cone, axis vertical and vertex down, contain a given volume of liquid, discuss the question of whether it can be emptied by rotation about a vertical axis.

(8) If the measured pressure and velocity at a given point in a pipe full of flowing water are respectively 10 pounds per square inch and 18 f. s.; what is the pressure head; what is the velocity head; what, supposing there is no friction, would be the hydrostatic head if the flow were stopped?

(9) At points A and B in an outflow pipe the hydrostatic heads are respectively 6′ and 5′, and the cross sectional areas are $1\frac{1}{4}$ and $2\frac{1}{4}$ square feet. If the pipe is discharging 16 cubic feet of water a second, what are the velocities, the velocity heads and the pressure heads at the two points? What is the pressure per

square inch at A and at B before the flow begins and after it is established?

(10) A vertical vessel of hour-glass shape, full of water, has a cross section of .4 square feet at its smallest part, which is 4 feet below the surface, and is discharging 8 cubic feet per second from an orifice near its bottom. What is the pressure in the water at the smallest section, and what would happen if a small tube connected that part with another vessel of water at a lower level?

(11) If the hydrostatic pressure in a pipe were 80 lbs. per square inch, what velocity must it have to reduce the pressure to 50 pounds per square inch? Water is flowing with a velocity of 25 f. s. in a pipe, how much will the pressure per square inch be increased if the flow is stopped?

(12) What is the velocity of efflux from a small orifice under a head of 6″; of 5′; of 100′?

(13) What would be the velocity of efflux from a small orifice into a vacuum under a head of 6″; of 5′; of 100′?

(14) What would be the velocity of efflux from a small orifice under a surface pressure of 100 pounds per square inch and a head of 6″; of 5′; of 100′?

(15) Water is being discharged through a small underwater orifice under a head of 60′ on one side and 10′ on the other. What is the velocity of efflux?

(16) What is the velocity with which water under a head of 250′ flows into a boiler in which the pressure is 40 lbs. per square inch by gauge?

(17) If the pressure in a boiler is 100 lbs. per square inch above the atmosphere, how fast will water flow through a small orifice 2′ below the water level?

(18) How fast will mercury flow out of a small orifice under a head of 3′ and a surface pressure of 10 atmospheres?

(19) A cylindrical vessel 3′ in diameter and 4′ deep, three-fourths full of water, is set rotating about its vertical axis until the water is just about to overflow. What will be the velocity of efflux from a small orifice in the base at a distance of 15″ from the axis?

(20) A circular tube of radius a, half full of fluid, is made to revolve uniformly about a vertical tangent. Show that the diameter through the open surfaces of the fluid is inclined at $\tan^{-1}\frac{\omega^2 a}{g}$ to the horizon.

(21) A circular tube of radius a, containing a filament of mercury which subtends an angle 2α at the center, rotates about a vertical diameter. Show that the filament will divide at its lowest point when the angular velocity reaches $\sqrt{\frac{g}{a}\sec\frac{\alpha}{2}}$.

(22) An open vertical cylinder, height h, radius a, full of liquid, rotates uniformly about its axis, the peripheral velocity being v. Show that the vertex of the free surface is $\frac{v^2}{2g}$ below the rim; that a volume $\frac{\pi a^2 v^2}{4g}$ of liquid is spilt, and that when the motion is stopped the surface will be $\frac{v^2}{4g}$ below the rim.

(23) Show that if, in the preceding example, $\frac{v^2}{2g} > h$, the bottom of the cylinder will be uncovered in a circle of area $\pi a^2\left(1 - \frac{2gh}{v^2}\right)$, and that when the cylinder is stopped the liquid will stand at a depth $\frac{2gh^2}{v^2}$.

(24) If the cylinder is closed by a piston of weight P, and the weight of the liquid is W, prove that the piston will rise when $\frac{v^2}{2g}$ exceeds $\frac{2hP}{W}$.

(25) At what angular velocity does the pressure at the lowest point of a closed sphere, full of liquid, and rotating about a vertical diameter, cease to be the greatest pressure on the surface?

(26) If a closed vessel full of fluid is rotating uniformly about an axis, prove that the pressure which results from the rotation upon any portion A of the enclosing surface is $\frac{\rho\omega^2}{2}Ak^2$, where ω is the angular velocity, ρ the density of the fluid, and Ak^2 the moment of inertia of the surface area A about the axis of rotation.

(27) A hollow cone, vertex up, just full of liquid revolves uniformly about a vertical element. Prove that the pressure on the base is $\frac{3W}{8}\left[\frac{a\omega^2}{g}(1 + 5\cos^2\alpha)\tan\alpha + 8\cos\alpha\right]$, where W is the weight of the liquid, 2α the vertical angle, and a the radius of the base.

CHAPTER VII.

Discharge through Small and Large Orifices.

(62) In order to compute the actual discharge of liquid through an orifice we must know both the velocity and the cross section of the jet at some point. We have seen that the theoretical velocity of efflux is given by the formula $v = \sqrt{2gh}$, and that under certain conditions this velocity is very nearly attained, not at the orifice itself, but in a contracted section of the jet just beyond it.

It has been found that the form of the orifice materially influences both the amount of the contraction of the jet and its velocity, and we will now consider what modifications of the formula for theoretic discharge experiment shows to be necessary.

(63) As already stated, the "coefficient of contraction," which will be called c_1, is the number by which the area of an orifice must be multiplied in order to find the area of the least cross section of the jet. Similarly, the "coefficient of velocity," designated by c_2, is the number by which the theoretical velocity of efflux must be multiplied to find the actual velocity at the least cross section of the jet. Lastly, the "coefficient of discharge," called c, is the number by which the theoretic discharge must be multiplied to find the actual discharge.

Thus, if a is the area of an orifice discharging under a head h, and a' the area of the contracted vein; and if v and q are the theoretical, and v' and Q the actual

velocity of efflux and quantity discharged in unit time; we have $a' = c_1 a$; $v' = c_2 v = c_2 \sqrt{2gh}$; $c = c_1 c_2$; and also $q = av = a\sqrt{2gh}$; $Q = c_1 c_2 av = ca\sqrt{2gh}$.

The determination of the value of the coefficient of discharge (c) is alone necessary for the practical purpose of measuring the flow of a liquid, but a separation of this coefficient into its two factors c_1 and c_2 is necessary for the discussion of questions regarding the energy of jets.

(64) An orifice so formed that the escaping liquid only touches its inner edge is termed a "standard orifice," and, on account of the regularity of the results given by such orifices, they are used, whenever practicable, for the measurement of water. The object to be attained by a "standard orifice" is to reduce the surface of contact of the jet with the vessel as nearly as possible to a line, and this result is attained, as shown in Fig. 18, either by making the orifice in a thin plate or by leaving the inner edge a sharp corner and beveling the outer edge of the orifice.

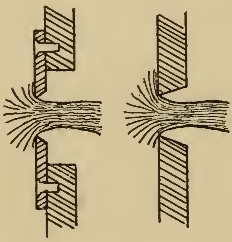


FIG. 18.

With circular orifices of this character the vena contracta, or section of least area of the jet, occurs at a distance from the plane of the orifice about equal to its radius. At this point a jet under steady flow looks like a clear crystal bar, while beyond it the jet gradually enlarges and its surface becomes more and more disturbed.

With standard orifices of rectangular shape a similar contraction takes place, the edges of the jet being angular and its cross section similar to the orifice until the vena contracta is passed.

Experiments with water flowing through standard orifices show that the coefficient of discharge decreases slightly with increase of head and also with increase of size of orifice, this being due to more perfect contraction of the jet, since the coefficient of velocity increases with the head, probably often exceeding 0.99. Furthermore, the coefficient of discharge is found to be slightly greater for squares than for circles of the same diameter, probably due to imperfect contraction at the corners; to be greater for rectangles than for squares of the same area; and to be very slightly less for submerged orifices than for orifices discharging into the air.

The mean value of the coefficient of contraction (c_1) is about 0.62, and the mean value of the coefficient of velocity (c_2) is about 0.98, making the mean value of the coefficient of discharge (c) about 0.61; and in all calculations of the flow of water through standard orifices these mean values can be used without any very material error.

(65) If a rectangular orifice has an edge at the bottom of the vessel, the liquid flowing through its lower portion moves in lines perpendicular to the plane of the orifice and so there will be no contraction on the lower side of the jet. So, too, if the orifice is in a corner of a rectangular vessel, the contraction will be suppressed on two sides of the jet, and, indeed, experiment shows that there is more or less suppression of contraction whenever an orifice of any shape

is close to the sides or bottom of the vessel. In such cases there is a somewhat increased discharge, but its amount varies greatly with the circumstances and is not well known, so that suppression of the contraction should be avoided where accurate results are sought.

If the inner edge of the orifice be rounded, the contraction of the jet is reduced, and the discharge increased; and, if the curve follows as nearly as possible the natural lines of contraction, as shown in Fig. 19, the issuing jet can be made to completely fill the orifice. This, however, is not really a suppression of the contraction, since the cross section at ab must bear about the same relation to that at cd as does the contracted vein to a standard orifice. With a carefully shaped mouthpiece of polished metal the mean value of the coefficient of velocity (c_2) is about 0.97, and, since there is no contraction, $c_1 = 1.00$; and the coefficient of discharge (c) is also 0.97.

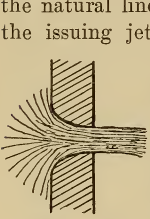


FIG. 19.

Generally speaking, the more the direction of the liquid flowing in from the sides differs from that of

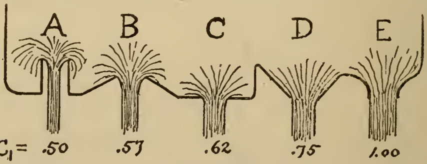


FIG. 20

traction in case A closely approaches the minimum
the main stream, the greater the contraction of the vein. Thus in Fig. 20, while the coefficient of con-

possible value 0.50; in case B it would be about 0.57; in case C about 0.62; in case D about 0.75; and in case E nearly 1.00.

(66) When water is allowed to discharge through a short tube, or ajutage, $2\frac{1}{2}$ or 3 times as long as it is wide, the stream is uncontracted and non-transparent, and, while the velocity of efflux is less, the discharge is considerably greater than in the case of a standard orifice of the same area. With a shorter tube, the vein does not touch the sides of the tube, and so it has no influence on the efflux. With a longer tube, the stream may not fill the tube at first, but if its outer end is closed for an instant so that the tube is once filled, it will continue to run full, giving what is called "discharge of a filled tube."

A "standard tube" is one of sufficient length to have the escaping fluid just fill its outer end, while its inner end is flush with the inner wall of the vessel and has a sharp edge like that of a standard orifice. With such a tube, since the jet fills the outer end, the coefficient of contraction (c_1) is 1.00, and so the coefficient of velocity equals that of discharge.

Experiment gives a mean value $c_2 = c = 0.82$ for the standard tube, showing that its discharge is $\frac{82}{61}$, or about 1.34 times as great as that of a standard orifice of the same cross section.

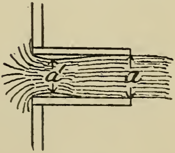


Fig. 21.

The use of a standard tube made of glass shows that the jet contracts in entering the tube just as it does in issuing into the air from a standard orifice, and, if a small hole be made in the tube abreast the contracted vein a′, in-

stead of water running out, air is sucked in. Furthermore, discharge with filled tube ceases if air is freely admitted to the tube at a′ (Fig. 21), and cannot be maintained under any circumstances when the head exceeds about 40 feet.

(67) The practical interest in the circumstances of discharge with short tubes lies in the light which it throws upon the subject of the entry of water from reservoirs into conducting pipes, but it has also a theoretical interest as exhibiting what at first sight appears to be a contradiction of the law of the conservation of energy. As stated, the coefficient of velocity at a′ (Fig. 21) is 0.82, and taking the coefficient of contraction at a′ to be .62, this shows that the velocity in the contracted vein is $\frac{82}{62}\sqrt{2gh} = 1.32\sqrt{2gh}$, so that the energy of the jet at the contracted vein is $\overline{(1.32)}^2 h = 1.74$ times the theoretical energy due to the head h. The explanation, however, is simple, and lies in the fact that the flow of water causes a partial vacuum around the contracted vein, so that the atmospheric pressure on the surface of the water in the reservoir acts to increase the flow. Thus, with the figures already given, we find that 1.74 times the real hydrostatic head is required to cause the actual flow in the contracted vein, which shows that there is an excess of pressure on the surface of the water in the reservoir over the pressure in the contracted vein equivalent to about three-fourths the hydrostatic head. Of course this excess of pressure could never in any case exceed the atmospheric pressure equivalent to 34 feet head, but even this is never attained since, when the hydrostatic head reaches about 40 feet, the reduction in velocity necessitated by the sudden expansion from

the contracted vein to a full mouth at atmospheric pressure could only occur with a negative pressure in the contracted vein, which is impossible. Consequently with a head of 40 feet or more a standard tube will not run full at the mouth.

Returning, then, to the question of energy, we see that while the energy of the jet at the contracted vein may exceed that due to the hydrostatic head, the excess must be expended in overcoming the atmospheric pressure in the outer part of the tube, so that in no case does the available exceed the theoretical energy. In fact the energy of a given quantity of water issuing in a jet depends solely upon its velocity of efflux, and since this velocity is 0.98 for a standard orifice and 0.82 for a standard tube, the energies, for the same weight of water, are as $\frac{\overline{0.98^2}}{0.82^2}$ or as .96 to .67 in favor of the orifice—4% of the total energy due to the head is lost with the orifice as against 33% lost with the tube. Even considering the energy of the jet per unit of time, regardless of the quantity of water used, we find a slight advantage on the side of the orifice, the two energies being about as .585 to .550.

We would naturally expect the greatest efficiency where the resistance to efflux is least, that is with a standard orifice; and experiment confirms this view, showing that with large heads there is practically no loss of energy when a standard orifice is used.

(68) The numerical values of coefficients thus far given are for water, and few experiments have been made to determine their values for other liquids. It may be said, however, that the more viscous a liquid the greater its coefficient of discharge. Thus a thick

oil flowing through a circular standard orifice under a moderate head has a coefficient of discharge of about 0.72; while mercury under the same conditions has a coefficient of discharge about 5% less than that of water.

(69) Hitherto we have considered only what are called small orifices—that is, orifices whose area is so small in comparison with the head that they can be assumed, without material error, to be under a uniform head equal to that on their centers. When an orifice is so large that there is considerable difference between the heads on its top and bottom, it becomes necessary to take account of the corresponding variations in the velocity of efflux. This is done by dividing the area into elementary strips and summing up the flows which would occur through them if they were separate small orifices. Of course this cannot give exact results, but it gives the closest approximation to the truth which can be mathematically determined.

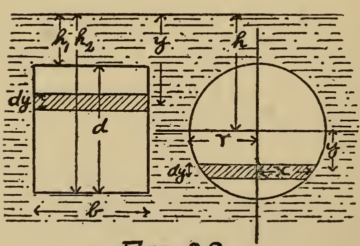


FIG. 22.

(70) As illustrations of the method of deducing formulæ for the discharge through orifices of large area we will take the two cases of rectangular and circular openings.

Referring to Fig. 22, if we consider the rectangle as made up of elementary strips, each dy high and at the

depth y, then the discharge through one of those strips, regarded as a small orifice, is $dQ = cbdy \times \sqrt{2gy}$, and so the discharge through the whole rectangle is $Q = \int_{h_1}^{h_2} cb\sqrt{2gy}\,dy$, the integration of which gives

$$(18) \qquad Q = \tfrac{2}{3}\,cb\sqrt{2g}\left(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}}\right)$$

in which c is the coefficient of discharge and may be taken to be 0.61 when there is full contraction.

Examination will show that this formula gives results slightly smaller than those given by the approximate formula $Q = cbd\sqrt{2gh}$, where h is the head on the center of the rectangle, but that the difference is only 0.3% when $h = 2d$ and only about 0.1% when $h = 3d$. Accordingly the exact formula (18) need only be used when the head on the center of the orifice is less than twice its vertical width.

Similarly, in the case of the circle, we have for the flow through an elementary strip, as shown in Fig. 22, $dQ = 2cxdy\sqrt{2g(h - y)} = 2c\sqrt{2gh}\ \sqrt{r^2 - y^2}\ \sqrt{1 - \frac{y}{h}}\,dy$. Whence $Q = 2c\sqrt{2gh}\int_{-r}^{r}(r^2 - y^2)^{\frac{1}{2}}\left(1 - \frac{y}{h}\right)^{\frac{1}{2}}dy$, or expanding $\left(1 - \frac{y}{h}\right)^{\frac{1}{2}}$ by the binomial theorem,

$$Q = 2c\sqrt{2gh}\int_{-r}^{r}(r^2 - y^2)^{\frac{1}{2}}\left(1 - \frac{y}{2h} + \frac{y^2}{8h^2} - -\right)dy,$$

each term of which is immediately integrable, giving

$$(19) \quad Q = c\pi r^2\sqrt{2gh}\left(1 - \frac{r^2}{32h^2} - \frac{5r^4}{1024h^4} - -\right),$$

in which again c is the coefficient of discharge, having a mean value 0.61.

Here again the approximate formula, $Q = c\pi r^2\sqrt{2gh}$, gives too great a result, but the error is only 0.4% when $h = 3r$, and so the exact formula need only be used when the head on the center of the circle is less than three times its radius.

(71) The principal practical application of formulæ for discharge through large orifices is to weirs, or notches in the top of the side of a reservoir, through which water flows. Only rectangular weirs will here be considered. Usually the vertical edges of the notch are so far from the sides of the reservoir or feeding canal that the sides of the issuing stream are fully contracted, and such a form is called a weir with end contractions, while the form in which the vertical edges coincide with the sides of the feeding canal is called a weir without end contractions.

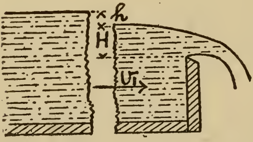


FIG. 23.

As shown in Fig. 23, the surface of the water takes a curved form in approaching the weir, so that the true head H must be measured some feet back from the crest, as the lower edge of the notch is called. For accurate results in the determination of the discharge it is also necessary that the crest and sides of the weir shall have definite sharp corners giving complete contraction.

Taking now our formula (18) for a large rectangular orifice, and putting $h_1 = o$, $h_2 = H$, we have

$$(20) \qquad Q = \tfrac{2}{3}cb\sqrt{2g}\,H^{\frac{3}{2}}.$$

In which the value of c, found by experiments, varies

from 0.65 to 0.58, being greater the less the head, and in the case of weirs with end contractions increasing with the value of b, which is called the length of the weir.

A mean value 0.61 may be used for contracted weirs and 0.63 for weirs with end contractions suppressed, but for accurate results the true value to suit the actual conditions must be obtained from a table of coefficients to be found in works on hydraulics.

The value of H may vary from 0.1 to 1.5 feet in practice, and the length of the weir from 0.5 to 20 feet and upwards.

(72) When there is an appreciable velocity of the water in the feeding canal at the point where H is measured, it must be taken account of, and this is done by regarding it as due to a fall h from the true level surface further upstream. The value of h is then taken to be $\frac{v_1^2}{2g}$, where v_1 is the mean velocity at the point where H is measured, this being called the "velocity of approach."

The best method of using this velocity of approach head in the formula for discharge through weirs is disputed. The most extensively used formula, that of Francis, takes h as head above the upper edge of the weir, H + h being the head on its crest, and, for weirs without end contractions, is

$$(21) \qquad Q = 3.33b[(H + h)^{\frac{3}{2}} - h^{\frac{3}{2}}]$$

in which 3.33 is a mean experimental value of $\frac{2}{3}c\sqrt{2g}$, and which is modified, for weirs with end contractions, by using b — 0.2 H instead of b for the length of the weir.

Smith's formula is

$$(22) \quad Q = \tfrac{2}{3} c b \sqrt{2g}(H + nh)^{\frac{3}{2}}$$

the value of c being given in his tables, and, as already stated, varying from 0.58 to 0.65, according to circumstances, and n being 1.4 for weirs with full contraction and 1.33 for weirs with end contractions suppressed.

Formula (21) is the more correct in form, while formula (22) takes account of the fact that the central current of the stream has a velocity greater than v_1, which is the mean velocity of approach. In good practice the velocity of approach should be as small as possible, not exceeding 1 f. s., and it is only occasionally, and in the case of weirs, that it needs to be considered.

(73) In the case of orifices, fortunately, the velocity of approach is almost invariably so small as to exert no appreciable effect on the discharge. Should it be desirable to allow for it, a consideration of the methods of deducing the formula $v^2 = 2gh$, both of which assume a zero velocity at the surface, indicates that where that velocity is not zero, but has a value v_1, we must put $v^2 = 2gh + v^2_1$. But if A be the cross section of the vessel at the surface, and a the orifice area, $v_1 = \frac{av}{A}$. Whence we have—

$$(23) \quad v = \sqrt{\frac{2gh}{1 - \left(\frac{a}{A}\right)^2}}.$$

Thus, as was to be expected, the discharge is least when there is no velocity of approach, and, as will be found upon trial, when the area of the orifice is $\frac{1}{20}$

the cross section of the vessel, the effect of the velocity of approach ceases to be appreciable.

(74) We are now in a position to calculate the quantity of water which will flow from any given orifice in a given time. It is only necessary to be careful in applying the formulæ to use the same units for all the factors. Thus if g is given the value 32, which is feet per second, h must be put in feet, and a in square feet, when Q will be found in cubic feet per second.

As an example, let us calculate how long it will take for a rectangular pontoon, 20′ × 10′ × 4′, which draws one foot when empty, to be sunk by the entry of water through a hole of 18 square inches area in its bottom. Here there is a constant difference of head of 3′ between the water without and that within the pontoon as the latter gradually settles, and, when $20 \times 10 \times 3 = 600$ cubic feet of water have entered, the reserve buoyancy is wholly exhausted, and the pontoon will sink. The influx per second is $Q = .61 \times \frac{18}{144} \sqrt{2 \times 32 \times 3} = 1.055$ cubic feet, and so it will take $\frac{600}{1.055} = 568.8$ seconds = 9 m. 48.8 sec. to sink.

PROBLEMS VII.

(1) How much water will flow through a standard opening 8 square inches in area, under a head of 20 feet, in 5 minutes?

(2) If 325 cubic inches of water per second flow through a standard opening of 5 square inches, what is the head?

(3) How many cubic feet of water will flow through

a standard orifice of one square inch in an hour, if the water level is maintained 9 feet above the orifice?

(4) Deduce a formula for the discharge through a triangular orifice, altitude a and base b, when the vertex is in the surface and the base horizontal.

(5) Deduce a formula for the discharge through a triangular orifice, altitude a and base b, when the base is in the surface.

(6) Deduce a formula for the discharge through a segment of a parabola, height a, base 2b, when the base is in the surface.

(7) Deduce a formula for the discharge through a segment of a parabola, height a, base 2b, when the vertex is in the surface and base horizontal.

(8) Calculate the discharge per minute through a rectangular orifice, 2 feet wide and 1 foot high, with its upper edge $1\frac{1}{2}$ feet below the surface.

(9) Calculate the discharge per minute through a circular orifice of 6″ diameter with its center 16″ below the surface.

(10) A rectangular pontoon $18' \times 9' \times 3'$, drawing 6″ when empty, has a hole of 9 square inches area in the bottom. How long will it take to sink?

(11) What will be the discharge per second through a circular orifice a foot in diameter with its upper edge at the surface?

(12) What will be the discharge through a triangular orifice of 18″ base and 6″ height, the base being in the surface?

(13) What will be the discharge through a triangular orifice of 6″ base and 18″ height, the vertex being up and at the surface and the base horizontal?

(14) A small stream is dammed with boards, through

which a rectangular hole 1′ wide and 6″ deep is cut. The water ceases to rise further when it stands 1′ above the top of the orifice. What is the flow of water in the stream?

(15) What is the discharge per second over a weir 6′ long, the head being 0.465 feet and the velocity of approach being negligible?

(16) Compare the discharges calculated by Francis's formula and by Smith's for a weir, with end contractions, 4′ long, the head being 0.427 feet and the velocity of approach 1 f. s., using $c = 0.61$.

(17) The floating dock at Algiers will support an 18,000-ton ship with the floor of the dock just awash, and has 2′ freeboard when carrying a 15,000-ton ship. How long does it take to sink it from the second to the first position by the influx of water through 16 circular valve openings, each of 16″ diameter, in the bottom of the dock, which is 17′ 6″ below its floor?

(18) The quantity of water discharged through a standard orifice of one square inch area under a head of 6″.5 is known as a "miner's inch." If the discharge through an orifice 250″ long by 4″ high, under a head of 9″ on its center, is sold for 1000 miner's inches, what error is made?

(19) What error is made in selling the discharge through a rectangular orifice 12″ deep and 12″.75 wide, with a head of 6″ above the upper edge, as 200 miner's inches?

(20) A hole of one square inch area is made in a boiler under a steam pressure of 180 pounds per square inch. What will be the approximate discharge of water per second?

(21) Water is discharging through a standard tube

under a head of 40′. What are the velocity and pressure heads at the mouth of the tube; what are they at the contracted vein; how do you account for the loss of head at each of these points?

(22) A vessel of 60 square inches cross section is kept full of water which is discharging through a circular orifice 5″ in diameter under a head of 8′. What is the velocity at the surface and what the discharge per second?

(23) In order to measure the flow of water in a canal 5′ wide, a transverse partition 2′ high, with sharp upper edge, is placed in it, and when steady flow is established, the height of the surface is found to be 1.′5 above the sill. How much water per second flows through the canal? (Use Smith's formula with $c = 0.64$ and $n = 1.33$.)

(24) An empty, air-tight, rectangular box, 8′ high, is placed with its top 12′ below the surface of the sea. If a small hole is made in its bottom, how high will the water rise in the box?

CHAPTER VIII.

Time of Emptying Vessels—Efflux of Gases—Flow Through Pipes.

(75) When a vessel, or reservoir, receives no inflow of water while it is being emptied through an orifice, the head is a dropping one, and not constant.

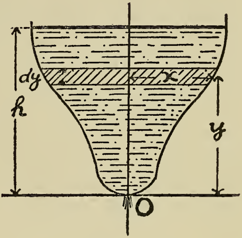


FIG. 24.

Let Fig. 24 represent a vertical section, through a small orifice at o, of a vessel of any shape. Take axes as shown; suppose the orifice area to be a; and let $K = f(y)$ be the surface area at the distance y above the origin. Then, if dy be the drop of the surface in the time dt, we must have $-\,Kdy = ca\sqrt{2gy}\,dt$, each of these expressions being the discharge in the time dt (dy is negative because y decreases as t increases). Hence we have $dt = \frac{-\,f(y)\,dy}{ca\sqrt{2gy}}$

and $\int_o^\tau dt = -\int_h^o \frac{f(y)dy}{ca\sqrt{2gy}} = \int_o^h \frac{f(y)dy}{ca\sqrt{2gy}}$, or

$$(18) \qquad \tau = \int_o^h \frac{f(y)dy}{ca\sqrt{2gy}}\,.$$

In which τ is the time corresponding to the fall of the surface from the height h to the orifice, or the time of emptying the vessel.

Thus, for example, take a cylindrical vessel, of height

h and constant cross section A, being emptied through an orifice of area a in the bottom. Then we have $\tau = \frac{A}{ca}\int_0^h \frac{dy}{\sqrt{2gy}} = \frac{2A}{ca}\sqrt{\frac{h}{2g}}$. This is twice the time it would take to discharge the same quantity under a constant head h, for in that case the discharge per second would be $ca\sqrt{2gh}$, and, the contents of the cylinder being Ah, the time to discharge its contents would be $\frac{Ah}{ca\sqrt{2gh}} = \frac{A}{ca}\sqrt{\frac{h}{2g}}$.

As another example, take a sphere of radius r, with a small orifice of area a at its lowest point. Here, again taking the origin at the orifice, we have for the variable surface area $K = \pi x^2 = \pi(2ry - y^2)$. Hence $\tau = \frac{\pi}{ca\sqrt{2g}}\int_0^{2r}\frac{2ry - y^2}{\sqrt{y}}dy = \frac{\pi}{ca\sqrt{2g}}\left[\frac{4ry^{\frac{3}{2}}}{3} - \frac{2y^{\frac{5}{2}}}{5}\right]_0^{2r}$ $= \frac{16\pi r^{\frac{5}{2}}}{15ca\sqrt{g}}$.

(76) If, on the other hand, it were desired to determine the time during which the surface will fall from the height h_1 to the height h_2, it is only necessary to integrate between those limits instead of between the limits h and o. Thus, in the case of a cylinder of cross section A, we have, for the time from height h_1 to height h_2 of water above the orifice, $t = \frac{2A}{ca\sqrt{2g}}(h_1^{\frac{1}{2}} - h_2^{\frac{1}{2}})$. Moreover, from this equation, by transposing and squaring, we get $Q = A(h_2 - h_1) = tca\left(\sqrt{2gh_2} - \frac{tcag}{2A}\right)$, where Q is the discharge in t seconds; whence the discharge in the first second is seen to be $ca\left(\sqrt{2gh_2} - \frac{cga}{2A}\right)$.

(77) The emptying and filling of a canal lock affords a practical example of the use of the formula for discharge under a variable head. The lock B (Fig. 25) is filled by opening one or more orifices in the upper gate, which separates it from the "head bay" A, and is emptied through orifices in the lower gate into the "tail bay" C. Consequently, both in filling and in emptying, the head on the orifices varies from H to o. Let the constant surface area of the lock be G, and the orifice area be a. Then the time to fill the lock is $-\frac{G}{ca}\int_{H}^{o}\frac{dy}{\sqrt{2gy}}$ and the time to empty the lock is $-\frac{G}{ca}\int_{H}^{o}\frac{dy'}{\sqrt{2gy'}}$, each of which integrals equals $\frac{2G}{ca}\sqrt{\frac{H}{2g}}$.

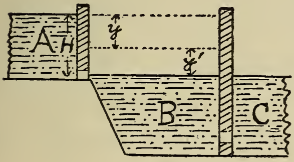


FIG. 25.

Thus if $G = 80' \times 20' = 1600$ square feet; $H = 12'$; and $a = 8$ square feet; we have $\tau = \frac{200}{c} = 322$ seconds $= 5$ min. 22 sec., if c has its value (0.62) for sharp cornered orifices. But if the inner edges of the orifices are rounded and c is increased to 0.80, we have the time of filling or emptying the lock reduced to 4 min. 10 sec.

(78) It was once customary to measure intervals of time by a water clock, or clepsydra, which consisted merely of a vessel from which water escaped through a small orifice, the fall of the surface indicating the

elapsed time. Assuming such a vessel to have the form of a surface of revolution about a vertical axis through the orifice, and taking x as the radius of a transverse section at the height y above the orifice, we have $\frac{\pi x^2 dy}{ca\sqrt{2gy}} = dt$, or $\frac{dy}{dt} \propto \frac{y^{\frac{1}{2}}}{x^2}$, whence we see that if a vertical section of the vessel has the equation $y = kx^4$, $\frac{dy}{dt}$ will be constant and so the surface will descend equal distances in equal times.

(79) The laws of efflux thus far set forth apply only to liquids, their theoretical deductions having assumed that the density of the fluid considered was constant, and their empirical modifications being based upon experiments with liquids. We will therefore briefly consider what modifications of those laws are necessary in order that they may apply to gases.

Returning then to the deduction of Bernoulli's theorem in Chapter VI; and supposing that a stream tube in a mass of gas, and not of liquid, in steady motion, is being considered, it will still be true that the same mass must enter the stream tube at P in a unit of time as leaves it as P_1 in the same interval,—we still have $\rho v\sigma = \rho_1 v_1 \sigma_1$, only now ρ and ρ_1 differ whereas in a liquid they are equal. This change of density of the gas due to its passage from P to P_1, however, introduces an essentially new element into the problem, since the gas, by virtue of its expansive power, possesses a so-called intrinsic energy which is a function of its density, and which therefore is greater or less at P_1 than at P depending upon whether it is more or less compressed there. Now the intrinsic energy of unit mass of any gas is $\int_0^p p\,dv$ where v is the

volume of unit mass and this, of course, is the same thing as $-\int_0^p p d\left(\frac{1}{\rho}\right)$. Hence in our statement of the equality of the energies flowing into and out of our imaginary stream tube, we must add to one side the intrinsic energy of the entering gas and to the other side that of the departing gas. Thus we have

$$\frac{p}{g\rho} + \frac{v^2}{2g} + z - \frac{1}{g}\int_0^p p d\left(\frac{1}{\rho}\right)$$
$$= \frac{p_1}{g\rho_1} + \frac{v_1^2}{2g} + z_1 - \frac{1}{g}\int_0^{p_1} p d\left(\frac{1}{\rho}\right).$$

Now, if we assume that the change of pressure from p to p_1 takes place without gain or loss of heat, or is an adiabatic change, we have $p = k'\rho^\gamma$, k' being a constant and γ the ratio of the specific heats of the gas, whence $-\int p d\left(\frac{1}{\rho}\right) = k'\int \rho^{\gamma-2} d\rho = \frac{k'\rho^{\gamma-1}}{\gamma-1} = \frac{p}{(\gamma-1)\rho}$.

This reduces our equation to

$$\frac{p}{g\rho} + \frac{v^2}{2g} + z + \frac{p}{g(\gamma-1)\rho}$$
$$= \frac{p_1}{g\rho_1} + \frac{v_1^2}{2g} + z_1 + \frac{p_1}{g(\gamma-1)\rho_1}, \text{ or}$$

$$\text{(19)} \quad \frac{\gamma p}{g\rho(\gamma-1)} + \frac{v^2}{2g} + z$$
$$= \frac{\gamma p_1}{g\rho_1(\gamma-1)} + \frac{v_1^2}{2g} + z_1 = C.$$

Thus we see that Bernoulli's theorem applies to gases subject to adiabatic changes with only the substitution of $\frac{\gamma}{\gamma-1} \cdot \frac{p}{g\rho}$ for the term $\frac{p}{g\rho}$.

(80) Applying this theorem, now, to the case of the efflux of a gas from a large reservoir at pressure p_1 into another where the pressure is p_2, through a small

orifice, we can consider v_1 as zero, and so get for the velocity of efflux, $\frac{v_2^2}{2g} = \frac{\gamma}{\gamma - 1}\left(\frac{p_1}{g\rho_1} - \frac{p_2}{g\rho_2}\right) + z_1 - z_2$. But $(z_1 - z_2)$, the difference of the heads at a point inside the vessel and at the orifice, is so small in comparison with the other part of the expression that it may be neglected. Moreover, since $p = k'\rho^\gamma$, we have $\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}}$, or $\rho_2 = \rho_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}}$. Hence we have, finally, $\frac{v_2^2}{2g} = \frac{\gamma p_1}{g\rho_1(\gamma - 1)}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]$, or, calling $\frac{\gamma}{\gamma - 1} = n$, and calling the velocity of efflux v,

$$(20) \quad v^2 = \frac{2gnp_1}{w_1}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{1}{n}}\right]$$

in which w_1 is the weight of unit volume of the gas at pressure p_1 and $n = \frac{1.408}{.408} = 3.541$, and which reduces for air to $v = 2413\sqrt{1 + \alpha t}\sqrt{1 - \left(\frac{p_2}{p_1}\right)^{.29}}$ f. s., or, for $t = 20°$ C., to $2510\sqrt{1 - \left(\frac{p_2}{p_1}\right)^{.29}}$ f. s.

(81) Of course (20) should reduce to the expression $v^2 = 2g\left(\frac{p_1 - p_2}{w_1}\right)$, corresponding to $v^2 = 2gh$ for liquids, when p_1 and p_2 are so nearly the same that the density of the gas within and without the vessel may be considered equal, and, to show that this is so, let $p_2 = p_1 - x$, where x is small compared with p_1. Then (20) becomes $v^2 = \frac{2gnp_1}{w_1}\left[1 - \left(1 - \frac{x}{p_1}\right)^{\frac{1}{n}}\right]$, and since $\left(1 - \frac{x}{p_1}\right)^{\frac{1}{n}} = 1 - \frac{x}{np_1} + \ldots$, we have, neglecting higher powers of the fraction $\frac{x}{p_1}$,

$$v^2 = \frac{2gx}{w_1} = 2g\left(\frac{p_1 - p_2}{w_1}\right).$$

This last formula may be used without great error whenever the difference of pressures is not more than one-fifth that of the reservoir. Thus if $p_1 = 18$ and $p_2 = 15$ pounds per square inch, and if the temperature be 25° C., then w_1, the weight of a cubic foot of air in the reservoir, is $\frac{18}{15} \times \frac{273}{298} \times .0807 = .0887$ pounds. Whence we have $v^2 = \frac{64 \times 144\ (18 - 15)}{.0887}$, and $v = \frac{96}{\sqrt{.0296}} = 558$ f. s. While the more accurate formula (20) gives, with the same data, a result about $3\frac{1}{2}\%$ greater.

(82) The coefficients of velocity and contraction for gases have not been well determined, and such experiments as have been made seem to indicate that they vary more widely than in the case of liquids. Weisbach recommends as coefficients of efflux for air, 0.56 for standard orifices; 0.75 for standard tubes; and 0.98 for well-rounded mouthpieces, and probably the coefficients determined for water can be used for air in all cases without very great error.

(83) We have assumed in the deduction of our formula for the velocity of efflux of gases that the pressure in the contracted vein is the same as that of the outside medium, and that the issuing jet has reached this reduced pressure adiabatically. Hence, to get the weight of gas discharged in unit time, we must multiply the volume discharged (cav, where c is the coefficient of efflux and a the area of the orifice) by w_2, the weight of unit volume at the outside pressure. But $w_2 = w_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}}$ when the change from p_1 to

p_2 is adiabatic, and so, calling $\frac{p_2}{p_1} = x$, we have for the weight of gas discharged in unit time—

$$(21) \quad W = caw_1 x^{\frac{1}{\gamma}} \sqrt{2ng\frac{p_1}{w_1}\left(1 - x^{\frac{1}{n}}\right)}.$$

Now, regarding this as a function of x, it will be found by putting $\frac{dW}{dx} = o$, that the value $x = \left(\frac{2n}{2n + \gamma}\right)^n = \left(\frac{2}{1 + \gamma}\right)^n = 0.527$ makes W a maximum. In other words, whatever value the pressure in the reservoir has, the maximum possible weight of gas will be discharged per second when the pressure of the outside medium is a little over half that in the reservoir.

It is remarkable that experiments seem to show that regardless of what the outside pressure may be, and even though the discharge be into a vacuum, the pressure in the contracted vein never falls below $0.527\ p_1$, so that, if the outside pressure is less than 0.527 times the inside pressure, the velocity of efflux assumes a constant value, found by putting $x = 0.527$ in (21), and which for air is $997\sqrt{1 + \alpha t}$ f. s., or at 20° C., 1036 f. s. Under the same conditions the weight of gas discharged per second is $kcaw_1$ pounds, where k is a constant having the value 520 for air.

(84) One of the most important practical problems in hydromechanics is the determination of the relation which exists between the dimensions of a conduit or pipe and the quantity of water which it will deliver under a given head.

We will first consider the simple case of a uniform pipe discharging with full section at its end. Let H

be the head which causes the flow, being the difference of level between the water surface in the reservoir and the center of the end of the pipe if the discharge is into the atmosphere, or the difference of level of the two water surfaces if the outlet is submerged. Let v be the velocity of discharge, which, since the cross section of the pipe is uniform, is also the velocity of flow throughout the pipe. Let d be the inside diameter of the pipe; and let l be its total length, following all its bendings. Then, if h_1 is the head which will produce the velocity v, and if h_2 is the total loss of head due to resistance in the pipe, we must have $H = h_1 + h_2$.

We have seen that under a head less than 40 feet a short tube placed just flush with the inner wall of a reservoir runs full at the mouth with a velocity $v = c\sqrt{2gh}$, whence we have $h = \frac{v^2}{2gc^2}$ for the head which will produce the velocity v in a pipe, the value of c being 0.82 for a sharp cornered entrance and approaching 1.00 for a well-rounded one. We shall assume c = 0.82 for all cases, and therefore have

$$(22)\quad h_1 = 1.5\frac{v^2}{2g}.$$

The pipe being of uniform diameter, the resistance to flow through it can only be due to bends, obstructions, and friction, and we will consider separately these three factors in the total loss of head in the pipe.

The loss of head caused by an easy curve is inappreciable, and even for a sharp curve is small, being stated by Weisbach to be only $0.3\frac{v^2}{2g}$ for a $90°$ change of direction of the axis of the pipe in a curve whose

radius equals the inner pipe diameter. So this part of h_2 need only be considered when there are a number of sharp bends, which should never happen in good practice.

The loss of head caused by a partly closed valve may be expressed in the form $k\frac{v^2}{2g}$, in which k, as given by Weisbach, is 5 or 6 for a valve about one-third closed and may reach several hundred if the valve is nearly shut. As our calculations are only to be applied to pipes running full flow, this cause of resistance will be considered non-existent, and it is only referred to as indicating the very serious loss of head which may result from an accidental obstruction in a pipe.

By far the greatest resistance to the flow of water in a long pipe is caused by friction against the walls and accompanying eddies and cross currents, and it is found by experiment that the loss of head from this cause is directly proportional to the square of the velocity of flow and the length of the pipe, and inversely proportional to the pipe's diameter.

We can therefore write for the loss of head due to all resistances in a uniform pipe of easy curves,

$$(23) \quad h_2 = \frac{f\,l}{d} \cdot \frac{v^2}{2g}$$

in which f is a coefficient depending for its value primarily upon the roughness of the inner pipe surface, but which is also found to diminish with increase of the velocity of flow and with increase of the diameter of the pipe.

Thus we have $H = \frac{v^2}{2g}\left(1.5 + \frac{fl}{d}\right)$, from which we get

$$(24) \quad v = \sqrt{\frac{2gH}{1.5 + \frac{f\,l}{d}}}$$

and this formula should be used for comparatively short pipes, v being first determined approximately by the use of an estimated value of f, and then more accurately by using the tabular value of f corresponding to the velocity of flow first found.

If the length of the pipe exceeds one thousand of its diameters, we may neglect the 1.5 in comparison with $\frac{f\,l}{d}$ and write

$$(25) \quad v = \sqrt{\frac{2gHd}{f\,l}}$$

(85) With clean iron pipes, either smooth or coated with coal-tar varnish, f varies from .01 for large pipes and high velocities to .04 for small pipes and low velocities. It may be assumed to be .03 for diameters from 0″.5 to 3″.0, and .02 for diameters from 1.′0 to 2.′0, for clean pipes and velocities from 3 f. s. to 6 f. s. For accurate work, however, its appropriate value must be sought in tables to be found in works on hydraulics.

With very foul pipes the values of f just given should be increased 50%, or even 75%.

(86) To determine the discharge through a long pipe we have $Q = \frac{\pi d^2}{4} v$, or

$$(26) \quad Q = \frac{\pi}{4} \sqrt{\frac{2gHd^5}{f\,l}}$$

and from (26) we find for the diameter of pipe which will discharge a given quantity of water (Q)

$$(27) \quad d = \left(\frac{8f\,lQ^2}{\pi^2 gH}\right)^{\frac{1}{5}}.$$

Also, for the head necessary to give the discharge Q through a given pipe,

$$(28) \quad H = \frac{8flQ^2}{\pi^2 gd^5}.$$

Thus we see that if the diameter of a pipe is doubled the quantity of water it will deliver is increased in the ratio $\sqrt{32} : 1$, or nearly sixfold; while the head required to furnish a given quantity of water through a pipe of given diameter is 32 times as great as if that diameter were doubled.

As a practical example, let us calculate the discharge of a pipe 1′ in diameter and 10,000′ long under a head of 100′. Taking $f = .02$, we have $v = \sqrt{\frac{64 \times 100 \times 1}{.02 \times 10000}} = \sqrt{32} = 5.65$ f. s. and $Q = \frac{\pi}{4} v = 4.44$ cubic feet per second. The head at the discharge end is therefore only $\frac{v^2}{2g} = 0.5$ feet, showing that 99.5 feet is the head used up in overcoming friction in the pipe.

(87) The formula for velocity in a long pipe, when put in the form $v = n\sqrt{\frac{d}{4} \cdot \frac{h}{l}}$ and written

$$(29) \qquad v = n\sqrt{R.\,S.}$$

is known as Chezy's formula, and is the one most used for water conduits of all sorts. In the Chezy formula R is called the "hydraulic radius," or the "hydraulic mean depth," and is found by dividing the cross section of the stream by the wetted perimeter of the conduit, this being $\frac{\frac{\pi d^2}{4}}{\pi d} = \frac{d}{4}$ for a full pipe of

circular section; while $s = \frac{h}{l}$ is called the "hydraulic inclination," h being the total head diminished by the head required to produce the velocity in the pipe $\left(H - \frac{v^2}{2gc^2}\right)$, and l being the total length of the pipe. The constant n is, of course, experimentally determined, and includes $\sqrt{2g}$, with g expressed in feet per second, so that in the use of Chezy's formula R must be expressed in feet.

For circular pipes, or semi-circular open conduits, having smooth inner surfaces, and for moderate velocities, n may be given the values 98, 114, 126, and 140 for diameters of 1″, 1′, 2′ and 4′ respectively.

As a practical example, we will find the discharge through a pipe 2″ in diameter and 96′ long under a head of 5′. Here, as an approximation, $v = 98 \sqrt{\frac{5}{24 \times 96}} = 4.56$ f. s.; from which we have for the head which will produce the velocity $h_1 = \frac{v^2}{2gc^2} = \frac{1.5 \times \overline{4.56}^2}{64} = 0.49$ feet. Therefore $H - h_1 = 4.'51$, and $v = 98 \sqrt{\frac{4.51}{24 \times 96}} = 4.34$ f. s.; whence $Q = \frac{\pi}{144} \times 4.34 = .095$ cubic feet per second.

With the same data formula (24) for short pipes gives $v = 4.13$ f. s. and $Q = 0.090$.

PROBLEMS VIII.

(1) Find the time of emptying a cone with vertical axis through a small orifice in its vertex.

(2) Find the time of emptying a cone with vertical axis through a small orifice in its base.

(3) Find the time of emptying a hemispherical bowl through a small orifice at its lowest point.

(4) A closed hemisphere full of water stands with its flat side down; how long will it take to empty it through a small hole in the base?

(5) A vessel formed by the revolution of a semi-cubical parabola ($ax^2 = y^3$) about its vertical axis is filled with water to the point where the radius of the surface equals its height from the vertex. How long will it take to empty through a small hole at the vertex?

(6) Water stands 20′ deep in a vertical cylinder of 50 square feet cross section. How long will it take for the surface to fall 10′ by reason of the discharge through an orifice of 9.5 square inches in its bottom; and how long will it take to then fall 5′ more?

(7) How long will it take to empty a sphere of 4′ radius through a hole of 1 square inch at its lowest point?

(8) The ditch around a fort is a mile long, 30′ wide, and 9′ deep, and is full of water. How long will it take to drain it to a depth of 3″ through a vertical cut 2′ wide? How long will it take to lower the surface another inch? (Assume opening to have well-rounded sides so that there is no contraction).

(9) A vessel is of the form given by the revolution of $y = 64x^4$ about the axis of Y, and has a circular orifice of 1″ diameter at its lowest point. What vertical fall of the surface of water contained in it will measure an elapsed time of one second?

(10) Two vessels of uniform horizontal cross sec-

tions, respectively A and B, communicate by a small submerged orifice of area a. If the levels differ by h feet, how long will it take for them to come to a common height?

(11) How long does it take to fill and how long to empty a single lock of mean dimensions 200′ × 24′, the filling orifice being 2.′5 wide and 4′ high and its center 5′ from both water surfaces when the lock is empty; and the emptying orifice being 2.′5 wide and 5′ high and entirely submerged? (Use $c = .62$).

(12) A large reservoir contains air at 120° C. and $17\frac{1}{2}$ pounds pressure per square inch. What will be the velocity of efflux into the atmosphere and what the volume and weight discharged per second through an orifice of 2 square inches?

(13) What is the velocity and weight per second of discharge of air at 22.5 pounds per square inch and 20° C. through an orifice of 1 square inch area into the atmosphere?

(14) A Whitehead torpedo flash contains 11.77 cubic feet of air at 1500 pounds per square inch. How long will it take for discharge through a circular hole of 2″ diameter to reduce the pressure to 50 pounds per square inch?

(15) A reservoir contains natural gas at 30 pounds per square inch and 20° C., weight of the gas being .0484 pounds per cubic foot at 0° C. and atmospheric pressure. What quantity and what weight will be discharged per second through a short tube of 4″ diameter into the atmosphere?

(16) A uniform pipe of 3″ diameter and 120′ long discharges water under a total head of 24′. What is the velocity in the pipe?

(17) What should be the diameter of a pipe to discharge 1200 gallons a minute at a point a mile distant from and 96 feet below the surface of a reservoir? (1 gal. = 231 cu. in.)

(18) What total head is needed for the discharge of 500 gallons a minute through a 6″ pipe a mile long?

(19) A pipe 400 feet long and 8″ in diameter discharges under a total head of 32 feet. What are the velocity and the discharge per minute if the pipe is straight and with no obstructions, and what are they if bends and a partly closed valve cause a loss of head of 4.5 times the velocity head?

(20) If the pipe of the preceding example were 4000 feet long, what would the velocity and discharge be under each of the two circumstances stated?

(21) What is the discharge in gallons per minute of a 1″ pipe 50′ long under a head of 12′?

(22) What must the average inclination of a pipe be in order that the velocity of flow may be $.82\sqrt{2gh}$ where h is the head on the entrance?

(23) What must be the diameter of a pipe 1000 feet long to discharge 3 gallons a minute under a head of 11.3 feet?

(24) A circular masonry conduit 4′ in diameter runs half full and has a slope of 1.′5 in 1000′. Compute its discharge by Chezy's formula, using n = 120.

(25) What must be the dimensions of a rectangular trough, of width double its depth, and with a slope of .002, in order that it may deliver 120 cubic feet of water a minute when running full, assuming n = 100?

(26) What should be the diameter of a conduit to deliver 12,000,000 gallons of water a day, the slope being .00016?

(27) A village street has 100 houses which draw water from a main 3000′ long which is supplied from a reservoir 100′ above the dead end of the main and a mile from its beginning. What must be the uniform diameter of supply pipe and main to be able to furnish water with a pressure head of 75′ at twice the average rate of consumption, which latter is 500 gallons a day for each house?

CHAPTER IX.

Compound Pipes—Water Power—Energy and Reaction of Jets.

(88) Thus far we have only considered uniform pipes, and changes of section must sometimes be taken account of.

If v_1 and v_2 are the velocities in adjacent pipe sections of areas respectively a_1 and a_2, and if there were no loss of head in passing from the section a_1 to the larger section a_2, the pressure head in a_2 would be greater than that in a_1 by the decrease of velocity head, or $\frac{v_1^2 - v_2^2}{2g}$. But actually the pressure in a_2 must exceed that in a_1 just enough to cause a retardation $v_1 - v_2$ of the mass of water $\rho a_1 v_1$ which enters it each second. Consequently the pressure intensity in a_2 must exceed that in a_1 by $\frac{\rho a_1 v_1 (v_1 - v_2)}{a_1} = \rho v_1(v_1 - v_2)$; and the head corresponding to this is $\frac{\rho v_1(v_1 - v_2)}{g\rho} = \frac{v_1(v_1 - v_2)}{g}$. Thus there has been a loss of head $\frac{v_1(v_1 - v_2)}{g} - \frac{v_1^2 - v_2^2}{2g} = \frac{(v_1 - v_2)^2}{2g}$, which, since $a_1 v_1 = a_2 v_2$, may be written in either of the following forms:—

$$(30) \quad h_2 = \left(\frac{a_2}{a_1} - 1\right)^2 \frac{v_2^2}{2g} = \left(1 - \frac{a_1}{a_2}\right)^2 \frac{v_1^2}{2g}.$$

It is this loss by sudden expansion which explains the fact, already mentioned, that a jet issuing from a

standard tube has so much less energy than one issuing from a standard orifice. In the second case the energy per unit weight of water is $c_2^2h = 0.96h$, since the coefficient of velocity $c_2 = 0.98$; while in the first case $c_2 = 0.82$, and the energy is 0.67h, or 0.29h less. Now as there is a loss of 0.04 with the standard orifice, we may assume an equal loss due to the contact of the jet with the mouth of the tube, and this leaves 0.25h to be accounted for by the expansion of the jet from its contracted section (0.62a) to the full section of the tube (a). But formula (30) shows this to be correct, since $\left(\frac{1}{.62} - 1\right)^2 \frac{v_2^2}{2g} = 0.375 \times .67h = 0.25h.$

(89) A sudden decrease in the section of a pipe causes a similar, but smaller loss of head due to the expansion of the stream after it has been contracted in passing from the large into the small section. Thus, if c_1 be the coefficient of contraction of the stream, and v_2 its velocity after reexpansion to fill the pipe, the loss of head, by formula (30), is given by

$$(31) \quad h_2^* = \left(\frac{1}{c_1} - 1\right)^2 \frac{v_2^2}{2g}$$

and, since the maximum value of c_1 would be 0.62, this loss cannot exceed $0.37 \frac{v_2^2}{2g}$. It should be noted, however, that v_2 in this case is the velocity in the small section of the pipe, which may be much greater than v_1, so that the loss of head may be many times the velocity head in the large section of the pipe.

(90) In good practice any necessary change of section in a pipe is made gradually, by means of a reducer, and does not cause any loss of head, but should there be, in any case, losses of head due to abrupt

changes of section, numerous bends, partially closed valves, or any other cause than friction, they must be taken account of by summing them up in the expression $k\frac{v^2}{2g}$ and then using the formula

$$(32)\quad v = \sqrt{\frac{2gH}{1.5 + \frac{fl}{d} + k}}.$$

(91) If, at intervals along a pipe running full flow, vertical tubes, open at each end, are placed with their lower ends flush with the upper inner surface of the pipe, the heights of the water surfaces in such pipes,

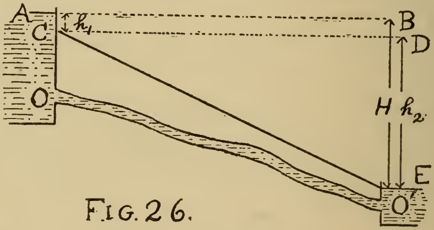


FIG. 26.

which are called open "piezometers," will evidently be the pressure heads at the corresponding points in the pipe, and a line joining these surfaces is called the "hydraulic grade line," or the "hydraulic gradient."

When the pipe is laid in an approximately straight line, and the only resistance is that of friction, the hydraulic gradient is also a straight line, its height above the pipe diminishing in direct proportion with the length of the pipe from its entrance. Thus, in Fig. 26, CE is the hydraulic gradient, beginning h_1 below the surface A and ending at the surface E. The

distance BE is the total head H, of which DE $= h_2$ is lost in overcoming resistances in the pipe, and BD $=$ AC $= h_1$ is the head required to produce the velocity of flow v, or $\frac{v^2}{2gc^2}$. At the entrance O the pressure head is OC, and at the outlet O′ it is O′E; while the velocity head throughout is $\frac{v^2}{2g}$, which is less than h_1 by the loss of head at entrance.

(92) It is necessary for continued full flow of a pipe that no part of it shall rise above the hydraulic grade line. Thus, if the entrance were above the point C, the head upon it would be less than h_1, and so would be insufficient to fill the pipe; and, if at any other point the pipe rises above CE, the pressure in it at that point will be less than the atmospheric pressure; air will collect there, and a great diminution of the discharge will result.

When a large part of a pipe conveying a liquid lies above the hydraulic gradient it is called a "siphon," and for the continued successful action of such a device it is necessary to furnish means, such as an air pump, for removing the air which will gradually accumulate at the point of greatest elevation.

(93) In the year 1890 about 6,000,000 H. P. was used for manufacturing purposes alone in the United States, of which 21.7% was derived from water and 79.3% from steam. As fuel becomes more expensive, however, the vast natural storehouses of water-power will surely be more and more drawn upon. It has been estimated that the rivers of the United States can supply some 200,000,000 H. P., and besides this there will always be available, while gravitation acts,

the power of the rising and falling tides. A simple calculation shows that a five-foot tide can furnish about 1800 H. P. from each square mile of ocean surface, and doubtless the time will come when efficient machinery will be devised to utilize this source of power.

(94) We will now consider briefly the methods by which the potential energy of a head of water is converted into useful work.

However water may be used to do work, its available energy is always the product of its weight and head. A quantity of water, say a cubic foot, caught in the bucket of a water wheel at the top of a fall, and descending through the vertical distance h feet, does the work 62.5h foot-pounds by the downward pressure of its weight. The same quantity of water, if allowed to enter a hydraulic press from the bottom of a reservoir in which the water stands h feet high will do the same amount of work, 62.5h foot-pounds, by virtue of its pressure, which is 62.5h pounds per square foot. And, finally, if the cubic foot of water be allowed either to fall freely h feet from the top of the reservoir or to issue in a jet from its bottom, its energy, or the work it is capable of doing, is still the same, being $\frac{62.5v^2}{2g} = 62.5h$ foot pounds.

In the first case energy of position, measured by the potential head h, is converted directly into work, and in the second case energy of position, measured by the pressure head $\frac{p}{w} = h$, is converted directly into work. In both cases the work is done by the static pressure exerted by the water while it descends,

and any acquired kinetic energy is wasted. The only difference is that in the water wheel one cubic foot of water descends h feet, and in the press h cubic feet of water descend one foot, the result being 62.5h foot-pounds of work done in each case.

In the third case energy of position has been transformed into kinetic energy, measured by the velocity head $\frac{v^2}{2g} = h$, and this has then to be converted into work by utilizing the dynamic pressure caused by the impulse of the jet.

Since there is no loss in the transmission of hydrostatic pressure, and since pressure head can be converted into velocity head with hardly any loss, the choice of method of utilizing a given water power depends upon the cost and efficiency of the different kinds of motors and the practicability of their use under the particular conditions of each case.

(95) When water under a head is used to do work by the direct action of its pressure, the energy is really only transmitted by the water and is not inherent in it. Of course it is elastically compressed and will do work if allowed to expand against a resistance, but this intrinsic energy, unlike that of a compressed gas, is always very small, however great the pressure, and a compressed liquid is only useful for transmitting energy, not as an independent source of it. It is convenient, however, to speak of the energy which water under pressure will transmit as the energy of the water itself, and in a sense this is true, since work can only be done when the water moves and so the amount of work done is really measured by the amount of water under pressure which is expended.

With this understanding, then, we say that the available energy per pound of water is h foot-pounds, where h is the pressure head in feet, and since $p = hw$, where p is the pressure per square foot and w the weight of a cubic foot of water, it is evident that the foot-pounds energy of water per cubic foot is its pounds pressure per square foot.

This may be made more directly apparent by considering the water to be doing work in a cylinder of one square foot cross section, when, evidently, the expenditure of one cubic foot of water under a pressure of p pounds per square foot will move the piston one foot, and so do p foot-pounds work.

Thus the energy per cubic foot of water delivered at a pressure of 750 pounds per square inch is $750 \times 144 = 108{,}000$ foot-pounds, and the use of a cubic foot of this water per minute in a motor of 80% efficiency would furnish $\frac{108000 \times .80}{33000} = 2.62$ H. P.

(96) Water is sometimes supplied under pressure in mains for use for elevators and other domestic motors, and, as an example of such a case, suppose it is desired to deliver 2.5 cubic feet of water per second, at 600 pounds pressure per square inch, through a 6″ pipe 2000 feet long, the distributing main being 18 feet above the pumping plant. Thus $v = \frac{4Q}{\pi d^2} = \frac{40}{\pi} = 12.73$ f. s., and $\frac{v^2}{2g} = 2.53$ feet; also the required pressure head is $\frac{600 \times 144}{62.5} = 1382$ feet. Therefore the work which the pump has to do is equivalent to lifting 2.5 cubic feet of water per second 1400 feet high and

at the same time overcoming the resistance in the supply pipe. The losses of head due to this resistance are as follows: (1) $h' = \frac{v^2}{2gc^2} = 1.5\frac{v^2}{2g}$ is used up in imparting the velocity v to the water in the pipe; (2) $h'' = \frac{f\,l}{d} \cdot \frac{v^2}{2g} = 80\frac{v^2}{2g}$ is used up in overcoming friction; (3) $h''' = m\frac{v^2}{2g}$, in which we will assume m = 4, is lost in passing through the pump. The total work of the pump must be, therefore, $2.5 \times 62.5[1400 + 2.53(1.5 + 80 + 4)] = 252{,}550$ foot-pounds per second = 459 H. P., and the available power in the distributing main is $\frac{2.5 \times 600 \times 144}{550} =$ 393 H. P., so that there has been a loss of nearly 14.5% up to the distributing point, the loss from that point on depending, of course, upon the various circumstances attending the use of the water by the consumers.

Calculation will show that the use of a pipe a foot in diameter would reduce the required horse-power of the pump to 400, making the loss less than 2%. Doubling the diameter of a pipe, however, requires for equal strength to withstand internal pressure a four-fold weight, which means approximately a four times greater cost, and this has to be weighed against the greater economy in operation which would result.

With such a system as just referred to an accumulator would be associated, to receive the surplus water at times of less than the average use, and to supply the deficiency at times of more than the average use.

(97) In the transmission of water every loss of head

not directly resulting from a change of level means the dissipation of useful energy in heat, with no possibility of its recovery, and the loss measured in foot-pounds is the product of the lost head and the weight of water delivered in pounds. It must be remembered, however, that the velocity head $\left(\frac{v^2}{2g}\right)$ enters as a factor into every loss of head, showing the great importance of keeping down the velocity when water-power is being transmitted, and also indicating that increased resistance to the flow, though it reduces the quantity of water discharged, may increase both energy and power.

The partial closing of a valve in a pipe may cause a loss of head several hundred times the velocity head, but the resulting decrease in the velocity head itself may make the actual loss very small, and, in fact, this loss becomes zero when the infinite resistance to flow of a completely shut valve has made the velocity zero. If the outlet of a pipe be completely closed the velocity head vanishes, the pipe becomes a part of the reservoir, and the pressure head equals the hydrostatic head, and by altering the size of the outlet the energy of the issuing stream may be made anything from an infinitesimal fraction to practically 100% of that corresponding to the hydrostatic head.

(98) Even with moderate velocities, the loss of head from friction is so great when water is carried through long pipes that for use as power it is almost always necessary to increase the energy of the jet by the use of an orifice of less size than the pipe itself, the loss of head from the contraction being reduced to almost nothing by a tapered mouthpiece or nozzle.

To illustrate the effect of this, we will consider the case of a fire engine which pumps water through 250′ of 2″.5 hose, the pressure at the engine being 125 pounds per square inch. If, then, the discharge be from the hose itself, we have $v = \sqrt{\dfrac{2gh}{1.5 + \dfrac{fl}{d}}} = 8\sqrt{\dfrac{288}{1.5 + 24}} =$ (taking $f = .02$) 26.9 f. s.; whence we see that of the total head of 288 feet, only 11.3 feet is utilized in velocity head, 276.7 feet head being used up in overcoming friction.

But now suppose a nozzle of one-third the diameter of the hose to be applied at its end, and let v_1 be the velocity of efflux, v being the velocity in the pipe. Then the losses, as before, are $.5\dfrac{v^2}{2g}$ at entrance and $\dfrac{fl}{d}\dfrac{v^2}{2g} = 24\dfrac{v^2}{2g}$ from friction, but the velocity head at the outlet is now $\dfrac{v_1^{\,2}}{2g}$, and, since $v_1 = 9v$, this equals $\dfrac{81v^2}{2g}$, and, moreover, the head to produce the velocity v_1 in a tapered nozzle is $1.04\dfrac{v_1^{\,2}}{2g}$: so we have for the total $(.5 + 24 + 1.04 \times 81)\dfrac{v^2}{2g} = h = 288$; whence $v =$ 13 f. s.; and $v_1 = 9v = 117$ f. s. The useful velocity head is now 213.9 feet and only 74.1 feet head has been lost.

The quantity of water discharged per second is diminished from 0.917 in the first case to 0.425 cubic feet in the second case, but the power of the jet is increased from 648 to 5553 foot-pounds per second, or nearly nine fold.

(99) Neglecting loss of head at entrance to the pipe, and assuming a nozzle of perfect form, so that there is no loss at exit, we have

$$(33)\quad v = \sqrt{\frac{2gh}{\frac{fl}{d} + \frac{d^4}{d_1^4}}},$$

where d is pipe diameter and d_1 nozzle diameter, and since the velocity of exit $v_1 = \frac{d^2}{d_1^2} v$, we have

$$(34)\quad v_1 = \sqrt{\frac{2ghd^5}{fld_1^4 + d^5}}.$$

Now the work per second, or power of the jet is $E = wQ\frac{v_1^2}{2g} = w\frac{\pi d_1^2}{4} \cdot \frac{v_1^3}{2g} = \frac{w\pi}{8g}\left(\frac{2ghd_1^{\frac{4}{3}}d^5}{fld_1^4 + d^5}\right)^{\frac{3}{2}}$; and by putting the derivative of E with respect to d_1 equal to zero it will be found that E has a maximum value when

$$(35)\quad d_1 = d\sqrt[4]{\frac{d}{2fl}}.$$

In the foregoing example, then, the maximum power would have been obtained by making the diameter of the nozzle $d_1 = d\sqrt[4]{\frac{5}{24 \times 2 \times .02 \times 250}} = \frac{d}{\sqrt[4]{48}} = 0''.95$, instead of $\frac{2.5}{3} = 0''.83$; and by calculation will be found to be about 6310 foot-pounds per second, or nearly 11.5 H. P.

(100) The distinction between the energy per unit weight of water, and the power, or energy per unit of time, of a stream, should be kept clearly in mind. The former is $\frac{v^2}{2g} = c_2^2 h$, (where c_2 is the coefficient of velocity), and measures the efficiency of the jet, which

efficiency, being the ratio of actual to theoretical energy, is $e = \frac{c_2^2 h}{h} = c_2^2$. The latter is the actual energy of the stream per unit of time, and so is $E = wQ\frac{v^2}{2g} = cwa\frac{v^3}{2g}$, where c is the coefficient of discharge and a the orifice area.

(101) When water flows from an orifice there must be a reaction of the jet equal and opposite to the force which gives it velocity. If, then, a be the area of an orifice with well-rounded entrance, under the head h, the velocity of efflux v is imparted in each second of time to a mass $\frac{awv}{g}$, and, since change of momentum in unit time is the measure of the force which causes it, the reaction of the jet must be $\frac{awv^2}{g}$, which, taking the coefficient of velocity to be unity, equals 2awh.

Hence the reaction of a jet, if we neglect loss of velocity by friction, is double the hydrostatic pressure on the orifice, and if, for example, the waters of Niagara could be received in the short arm of a vertical J shaped tube, they would balance a column twice as high as their 162-foot fall.

(102) As an example of the utilization of the reaction of a stream of water for doing work, we will consider the "Jet Propeller."

The method of jet propulsion consists of taking water into a vessel and, by means of a pump, ejecting it sternwards, the reaction thus produced driving the vessel forwards. Let u be the velocity of the vessel and v the sternward velocity, relative to the vessel,

imparted to W pounds of water by the pump each second. Then each second W pounds of water is accelerated from rest to an absolute velocity $v - u$, and so the reaction, or propelling force, is $\frac{W}{g}(v - u)$, and the work of propulsion in foot-pounds per second is

$$(36)\quad E = \frac{Wu}{g}(v - u).$$

As another way of looking at it, consider that the water, entering with the velocity u relative to the pump leaves it with the relative velocity v; whence the whole useful work of the pump is $\frac{W}{2g}(v^2 - u^2)$. But the water leaves the vessel with the absolute velocity $v - u$, and so carries off the energy $\frac{W}{2g}(v - u)^2$. Therefore the work used in propulsion is the difference of these, which is $\frac{Wu}{g}(v - u)$ as before.

The efficiency of the apparatus is, of course, found by dividing the useful work by the total work, and is therefore $\dfrac{\frac{Wu}{g}(v - u)}{\frac{W}{2g}(v^2 - u^2)}$, so that we have

$$(37)\quad e = \frac{2u}{v + u}$$

and this attains the value unity only when $u = v$, which could only happen if W were infinite.

It will be observed that the foregoing is all based upon the supposition that the water enters with a sternward velocity u relative to the ship, and that the pump accelerates it to v. Actually the water in

entering acquires part, and sometimes the entire forward velocity of the ship, in which latter case the energy $\frac{Wu^2}{2g}$ is lost; the total work of the pump is $\frac{Wv^2}{2g}$; the useful work is $\frac{Wu}{g}(v - u)$ as before; and the efficiency is $\frac{2u(v-u)}{v^2} = 2\frac{u}{v}\left(1 - \frac{u}{v}\right)$, which has a maximum value 0.50 when $u = \frac{v}{2}$.

The method by which the water is conducted through the vessel determines which of the foregoing values of e is most nearly correct. The value e = 0.71 has been attained in an experimental boat, but the low efficiency of the centrifugal pump which forms part of the apparatus makes jet propulsion practically much less efficient than screw propulsion.

Since the resistance is approximately fAu^2, where f is a coefficient of friction and A is the wetted surface of the vessel, we can put $fAu^2 = \frac{W}{g}(v - u)$ and to obtain an approximate value of the speed when we know W and v.

PROBLEMS IX.

(1) A pipe 6000′ long and 1′ in diameter leaves a reservoir 100′ below its surface and runs horizontal for 1000′; the next 1000′ inclines upwards till it is 27′ above the first level; the next 1000′ has a uniform downward slope of 1 in 20; the next 1000′ of 1 in 10; and the last 2000′ of 1 in 100. What is the velocity and what the pressure per square inch at the beginning of each 1000′ length? If the outlet be closed what

does the pressure per square inch at the same points become?

(2) Water is flowing 16 f. s. in a pipe of 4″ diameter; what is the loss of head due to a sudden change of diameter of the pipe to 8″? If the pressure head in the 4″ pipe at the entrance to the 8″ pipe is 20′, what is the pressure head in the 8″ section?

(3) If water is flowing 4 f. s. in a pipe of 8″ diameter, what is the loss of head due to a sudden change of diameter of the pipe to 4″? If the pressure head in the 8″ pipe is 20′, what is it in the 4″ pipe?

(4) If the power of the rising and falling tide could be all utilized, what horse-power could be obtained per mile surface at a place where the mean rise and fall was 3 feet?

(5) How much water at 400 pounds per square inch must be used in a motor of 75% efficiency to develop 12 H. P.?

(6) It is required to furnish 100 cubic feet a minute of water at 500 pounds per square inch in a main 50 feet above a pumping station. If the water is supplied through a mile of pipe of one foot diameter, what must the power of the pump be, and what is the percentage lost in delivering the water?

(7) What horse-power must a pump have to deliver 2,000,000 gallons of water per day through an 8″ pipe 1500 feet long at a station 220 feet above the pump? What part of the power is lost in overcoming resistances?

(8) What should the diameter of nozzle be for maximum power in the case of a 6″ pipe, 400 feet long, with a total head of 200 feet, and what is that power?

(9) What horse-power is furnished by discharge

through a 3″ pipe 800 feet long, with a total head of 400 feet, and what will it be if a nozzle 0″.82 in diameter be used?

(10) What diameter of nozzle should be used to get the most power from a 300-foot head discharging through 1200 feet of 1-foot pipe, and what horsepower will be furnished?

(11) A pump of 100 effective H. P. discharges directly into a stand pipe of 20 feet diameter from which 18,000 cubic feet per hour of water is being drawn. How long will it take to raise the water level from 80 to 100 feet?

(12) If a jet is an inch in diameter, what must its discharge be to make the reaction 200 pounds?

(13) What is the reaction of the jet from a nozzle 2″ in diameter with a pressure head of 200 feet at the nozzle entrance?

(14) The Niagara Power Company has 21 double turbines, each developing 5000 H. P. under a head of water of 136′. If their average efficiency is 0.75, how many cubic feet of water per second must be drawn from the falls for their use?

(15) If the resistance in pounds to the motion of a vessel is 4 times the square of her speed in f. s., what will be her speed when propelled by the reaction of a jet from a 2″ nozzle under a pressure of 150 pounds per square inch? What is the efficiency of this propeller?

(16) What must be the effective H. P. of the pumps to give 8 knots speed to a vessel whose resistance is $5v^2$ pounds at a speed of v f. s. by means of a jet from a 2″ nozzle, and how much water must be pumped through the vessel per second?

(17) An experimental torpedo-boat of 13 tons displacement, built by Thorneycroft in 1881, was given a speed of 12.6 knots by the discharge of a ton of water per second at 37.25 f. s. The I. H. P. was 170 and the efficiency of the engine 0.77, what was the efficiency of the pump; what was the efficiency of the jet propeller; and what does the latter value indicate as to the method of taking in the water?

(18) The Water-witch, an experimental jet propeller of 1160 tons made 9.3 knots with 760 H. P. Her water supply pipes were so arranged that the water practically attained the ship's speed before being expelled rearward. The efficiency of her engine was 0.70; that of her pumps 0.45; and the escape orifices aggregated $5\frac{1}{3}$ square feet. How much water did she discharge per second; at what velocity; and what was the total efficiency of her propelling machinery?

CHAPTER X.

Reaction and Impulse—Water Motors.

(103) The so-called Barker's Mill is another example of a pure reaction motor. This consists of a hollow drum kept full of water under pressure, and having hollow radial arms from the outer ends of which the water issues in jets tangentially to the motion of rotation caused by their reaction (Fig. 27). Suppose the pressure head h to be maintained in the drum by the influx of water through a pipe so large in relation to the efflux that the velocity of approach may be neglected, and let u be the linear velocity of the orifices, v the relative velocity of efflux, and W the weight of water discharged per second. Then the total energy supplied per second is Wh; and the effluent water carries off in the same time the energy $\frac{W}{2g}(v - u)^2$; and the difference, $W\left(h - \frac{(v-u)^2}{2g}\right)$, must be the work done on the wheel. But $v = \sqrt{2gh + u^2}$, since the total head on the orifices is the static head (h) plus the head due to rotation $\left(\frac{u^2}{2g}\right)$. Therefore the useful work is $\frac{W}{g}(uv - u^2)$ or,

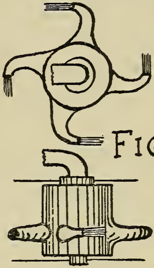


FIG. 27.

$$(38) \quad E = \frac{Wu}{g}(v - u).$$

This seems to show that the reaction of the jets is $\frac{W}{g}(v - u)$, whereas the absolute change in velocity, being from u in one direction to v — u in the other, is really v, and so the reaction must be $\frac{Wv}{g}$. The explanation of this anomaly is that there is a constant force opposed to the reaction of the jets, due to the fact that each second the weight of water W, entering the drum without velocity, must travel to the orifices, and in so doing acquire the velocity u and the energy $\frac{Wu^2}{2g}$. But the distance it travels per second in the line of action of the force giving it rotation is only $\frac{u}{2}$, and so the force constantly required to keep up the rotation of the water is $\frac{Wu^2}{2g} \div \frac{u}{2} = \frac{Wu}{g}$, and this must be subtracted from the reaction of the jets $\frac{Wv}{g}$ to get the effective rotating force $\frac{W}{g}(v - u)$.

(104) The efficiency of the Barker's Mill is the useful work $\frac{Wu\ (v - u)}{g}$ divided by the total work $Wh = \frac{W}{2g}(v^2 - u^2)$, and is therefore the same as that of the jet propeller,

$$(39) \quad e = \frac{2u}{v + u}.$$

Since $v = \sqrt{2gh + u^2}$, u can never be as great as v, but the efficiency approaches unity as u increases in

value, and could $u = \infty$ we would have $v = \infty$ also and so $e = 1$. The weight of water discharged per second, $W = wav$ (a = orifice area), would also be infinite; the rotating pressure, $\frac{wav}{g}(v - u) = \frac{wav}{g}(v - \sqrt{v^2 - 2gh})$, evaluated for $v = \infty$ will be found to be wah, the same as the static pressure due to the head alone; and the work of the wheel, being u times this, would be infinite.

In practice there is some reduction in the value of v from friction, and as speed increases the resistance of the air becomes important, so that the simple reaction wheel is no longer used as an hydraulic motor on account of its low efficiency.

(105) Just as it requires pressure to give water velocity, so pressure is exerted by moving water against any surface which either changes its direction or checks its velocity. Such pressures are called "dynamic pressures," and, like other impulsive forces, they are measured by the change in momentum which they produce in unit time.

If the jets issuing from a Barker's Mill are caused to impinge on surfaces rigidly secured to the mill itself, it will be found that when those surfaces are planes normal to the jets there is no rotation of the mill; when they are convex to the jets rotation in the usual direction occurs; and when they are concave to the jets there is rotation in the opposite direction. This shows that the dynamic pressure is greater the greater the angle at which the impinging jet is deflected.

When a surface completely destroys the velocity of

a jet in its original direction, all the water being deflected 90°, the dynamic pressure in that original direction is called the "impulse of the jet," and, as shown by the experiment just referred to, equals the reaction of the jet $\frac{Wv}{g}$.

(106) We shall now show that the total dynamic pressure in a given direction caused by a stream which impinges upon a smooth surface, passes over it, and then leaves it, is measured by the difference between the components, in the given direction, of the impulse of the approaching and the reaction of the departing stream, and is independent of the form of the surface.

Let AB, Fig. 28, be a horizontal section of a smooth vertical vane, and suppose that a weight of water W per second enters it with uniform velocity v at A and leaves it at B. It is then required to find the total dynamic pressure on the vane in the direction indicated by the arrows marked u. The stream is to be prevented from spreading, so that all the water will leave at B at the same velocity as it had at A, and at first we will suppose that it enters tangentially at A.

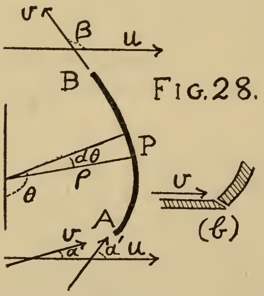


FIG. 28.

Calling s the length of the path AB, ρ its radius of curvature at any point P, and θ the angle that radius makes with a normal to the direction u, the mass of water on the vane is

always $\frac{Ws}{gv}$, and the mass on the element $\rho d\theta$ is $\frac{Ws}{gv} \cdot \frac{\rho d\theta}{s} = \frac{W\rho d\theta}{gv}$. There being no friction, the only force acting on this mass is the centrifugal force $\frac{W\rho d\theta}{gv} \cdot \frac{v^2}{\rho} = \frac{Wvd\theta}{g}$, and this must be the pressure on the element $\rho d\theta$ and acting normally to it. The component of this pressure in the direction u is $\frac{Wv}{g}$ sin $\theta d\theta$, and therefore the total pressure in that direction is $\int_{\alpha'}^{\beta} \frac{Wv \sin \theta d\theta}{g} = \frac{Wv}{g} (\cos \alpha' - \cos \beta)$. If now the stream approaches, not tangentially, but at the angle α with u, we must suppose it to be guided by a rim (as shown at (b), Fig. 28) in order that it may all enter the surface, and, however abrupt the change of direction at A may be, we can consider this rim as forming part of the vane and extend the integration to include it. So we have for the total pressure in the given direction, in all cases,

$$(40) \quad F_u = \frac{Wv}{g} (\cos \alpha - \cos \beta) .$$

And, since $\frac{Wv}{g}$ is the impulse of the approaching stream, and $\frac{Wv}{g} \cos \alpha$ its component in the given direction, and since, similarly, $\frac{Wv}{g} \cos \beta$ is the component in the given direction of the reaction of the departing stream, our proposition is proved.

Of course, when β exceeds 90°, as in Fig. 28, its cosine is negative, and so the reaction of the departing is added to the impulse of the approaching stream.

(107) If the stream does not enter nearly tangentially, there will be some loss of energy caused by eddies, foam, and changes of section, but neglecting this, the results of experiments fully accord with the foregoing theoretical deduction. For consider a surface of revolution with an impinging jet as shown at (a), Fig. 29. Then, since each element of the stream enters tangentially, our mathematical deduction applies rigorously, and the dynamic pressure in the direction of the jet is $-\frac{Wv}{g}(\cos 0° - \cos 90°) = -\frac{Wv}{g}$. But a simple weighing arrangement, when a flat plate

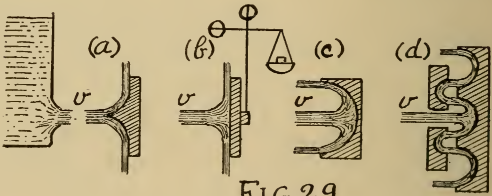


FIG. 29.

is used, as shown at (b), gives this same result, within about 4%, thus justifying our assumption that provided the stream enters the surface it is the direction of the approaching jet, and not that of the surface, which determines the impulse.

If then we apply our formula (40) to determine the pressure in the direction of the jet with a surface such as is shown at (c), we have $\alpha = 0$ and $\beta = 180°$, whence $F_v = \frac{2Wv}{g}$. With the arrangement shown at (d), where the jet suffers a double reversal of direction,

we should have theoretically $F_v = \frac{4Wv}{g}$, and experimentally the value $\frac{3.32Wv}{g}$ has been found, the loss from friction, etc., being in this case naturally large.

(108) The total action, then, of a jet impinging upon any smooth surface is equivalent to the action of two forces, one the impulse of the approaching stream acting at the entrance, the other the reaction of the departing stream acting at the exit.

If we wish to determine the amount and direction of the resultant of the total action of a jet on a surface, we have only to combine the aforesaid impulse and reaction as we would combine any two forces to find their resultant.

If we wish to determine the pressure in any direction caused by the action of the jet upon the surface, we have only to take the algebraic sum of the components, in that direction, of the aforesaid impulse and reaction.

In fact, the conclusion we have just come to amounts to nothing more than this: that the change of momentum in any direction per second measures the force acting in that direction,—for the momentum per second in the direction u was originally $\frac{Wv}{g} \cos \alpha$, and finally it is $\frac{Wv}{g} \cos \beta$ in the same direction, and so there must have been a force $\frac{Wv}{g}(\cos \alpha - \cos \beta)$ acting on the water in the direction opposite to u, and, since the same change of momentum takes place in each second, there must be a continuous pressure $\frac{Wv}{g}(\cos \alpha - \cos \beta)$ acting to produce it.

(109) In order, however, that work may be done by the pressure on a vane, the vane must move. Let u be the uniform linear velocity of the vane caused by the dynamic pressure F_u in the same direction. Then evidently the pressure depends upon the relative velocity of the water in contact with the surface, and formula (40), deduced for the case of a stationary vane, will apply to this case provided merely we use the relative velocity $V = v^2 + u^2 - 2uv \cos \alpha$ instead of the absolute velocity v (Fig. 24).

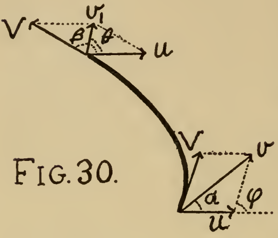


FIG. 30.

Therefore we have, in this case, $F_u = \frac{WV}{g} (\cos \varphi - \cos \beta)$, and since $V \sin \varphi = v \sin \alpha$, $V^2 \cos^2 \varphi = V^2 - v^2 \sin^2 \alpha = (v \cos \alpha - u)^2$, we get

$$(41) \quad F_u = \frac{W}{g}(v \cos \alpha - u - V \cos \beta).$$

Moreover, as Fig. 30 shows, $u + V \cos \beta = v_1 \cos \theta$ and so we have also

$$(42) \quad F_u = \frac{W}{g}(v \cos \alpha - v_1 \cos \theta).$$

Formula (41) shows that the total pressure in the direction u is the difference between the impulse and the reaction due to the *relative* velocities in the given direction of the approaching and departing streams; while formula (42) shows that the total pressure is also equal to the difference between the impulse and

the reaction due to the *absolute* velocities in the given direction of the approaching and departing streams.

(110) The work done on the vane per second, then, is either of the expressions—

$$(43) \quad E = \frac{Wu}{g}(v \cos \alpha - u - V \cos \beta)$$
$$= \frac{Wu}{g}(v \cos \alpha - v_1 \cos \theta),$$

and a consideration of the energies of the entering and departing streams will give the same results. For the absolute velocity of the jet before it touches the vane is v, and when it leaves the vane is $v_1 = \sqrt{V^2 + u^2 + 2uV \cos \beta}$, and so the work done on the vane, neglecting friction, must be the difference of the corresponding energies, or $\frac{W}{2g}(v^2 - v_1^2) = \frac{W}{2g}(v^2 - V^2 - u^2 - 2uV \cos \beta) = \frac{Wu}{g}$ $(v \cos \alpha - u - V \cos \beta)$ as before (since $V^2 = u^2 + v^2 - uv \cos \alpha$, or $uv \cos \alpha = v^2 + u^2 - V^2$).

Either of the two foregoing methods may be used to determine the power of a hydraulic motor which is actuated by the dynamic pressure of an impinging stream, and though we have assumed all parts of our vane to have the same linear velocity, which is not the case in practice, it is still always true that the work done, neglecting frictional losses, is the difference between the energy of the entering and that of the departing stream; and is, equally, the difference of the works done by the impulse of the entering and the reaction of the leaving stream. It is either of the expressions,

$$(44) \quad E = \frac{W}{2g}(v^2 - v_1^2) = \frac{W}{g}(uv \cos \alpha - u_1 v_1 \cos \theta)$$

in which v and v_1 are the absolute velocities of the

stream in entering and leaving; u and u_1 the velocities of the two parts of the vane where the stream enters and leaves it; and α and θ the angles respectively between v and u and between v_1 and u_1.

(111) Water wheels driven by the impulse of a jet consist of a number of vanes which turn about an axis, and come successively in front of the jet so that such portions of the water as do not strike a vane just leaving the jet are utilized by another vane just coming into position, and we may thus properly consider such a motor as consisting of a single vane against which the entire weight of water discharged per second (W) is being constantly delivered.

Considering the expression for the work of a wheel, $\frac{W}{2g}(v^2 - v_1^2)$, it is evident that for maximum efficiency there must be as little loss of energy from foam and friction as possible, and the absolute velocity of exit must be zero, which conditions are summed up in the statement that "the water must enter the wheel without shock and leave without velocity."

The first condition is met by so shaping the vanes that the relative motion of the entering stream is tangential to the surface it first impinges upon, which requires the relation $\frac{u}{v} = \frac{\sin(\varphi - \alpha)}{\sin \varphi}$ (Fig. 30), α being called the approach angle and φ the entrance angle.

The second condition requires the exit angle β to be 180°, and also that v_1 shall equal u_1, which would make $V = u$ and so $2\varphi = \alpha$; but β cannot have this value, since if it did the water would not leave the wheel, and so the best that can be done practically is

to make β from 150° to 175° and $\alpha = \frac{\varphi}{2}$ This gives $v = 2u \cos \alpha$, and $v_1 = 2u_1 \cos \frac{\beta}{2}$. But the efficiency of the wheel is $\frac{\frac{W}{2g}(v^2 - v_1^2)}{\frac{W}{2g}\, v^2} = 1 - \frac{v_1^2}{v^2}$, therefore

$$(45) \quad e = 1 - \frac{u_1^2 \cos^2 \frac{\beta}{2}}{u^2 \cos^2 \alpha} = 1 - \frac{r_1^2 \cos^2 \frac{\beta}{2}}{r^2 \cos^2 \alpha}$$

in which r and r_1 are the radii of the inner and outer edges of the vane.

Formula (45) indicates that α should be as small as possible, but it must be large enough to make the water enter the wheel, and its size is less important than is that of β.

The equation $v = 2u \cos \alpha$, which results from the condition $\varphi = 2\alpha$, shows that u should equal $\frac{v}{2 \cos \alpha}$ for the maximum efficiency, so that the smaller α the nearer u should approach $\frac{v}{2}$.

(112) As an example of the foregoing formulæ, we will determine the proper speed, the power, and the efficiency of a wheel, having the following characteristics: $r = 3'$; $r_1 = 4'$; $\alpha = 30°$; $\varphi = 60°$; $\beta = 150°$; and driven by the jet from a fixed nozzle,. v being 100 f. s. and W = 125 pounds. Then to avoid shock at entrance $u = \frac{v}{2 \cos \alpha} = 57.7$ f. s., and so the wheel should be adjusted to make 3.6 turns a second, or 216 turns a minute. But, since $u_1 = \frac{4u}{3} = 76.9$ f. s., $v_1 = 2u_1$

$\cos \frac{\beta}{2} = 39.8$ f. s. Therefore the work per second of the wheel is $\frac{W}{2g}(v^2 - v_1^2) = 16{,}426$ foot-pounds $= 29.9$ H.P.; and the efficiency is $1 - \frac{v_1^2}{v^2} = 0.841$.

(113) A very simple, and yet a very efficient water motor is a vertical impulse wheel specially designed for use with comparatively small quantities of water under great heads, and known in California, where it is much used, as a "hurdy-gurdy" wheel. It is an iron wheel, of diameter from 1 to 6 feet, according to the power, carrying on its outer rim a number of cups into which the jet from a fixed nozzle plays as they successively come into position in front of it. Each cup has a projecting sharp edge or rib across its center, to split the impinging jet, and is so formed that the direction of the stream is almost completely reversed after impact. It will easily be seen that the efficiency of such a wheel is greatest when the speed of the cups is half the velocity of the jet, since in that case the absolute velocity of the water after impact, provided the shape of the cups was such as to exactly reverse the motion of the water, would be zero, and the efficiency would then be unity, all the energy of the jet being transmitted to the wheel. In actual practice these wheels attain an efficiency of 0.85 and sometimes even more.

PROBLEMS X.

(1) What is the efficiency of a Barker's mill when the linear velocity of the orifices (u) is half the velocity of efflux (v), and what is the power under such condi-

tions if the head on the wheel is 25′ and the orifice area is 4 square inches?

(2) The linear velocity of the orifices of a Barker's mill is $\sqrt{2gh}$, where h is the head on the mill. What is the power and the efficiency?

(3) If the velocity of efflux of a Barker's mill is only 0.94 its theoretical value ($c_2 = 0.94$), show that the most advantageous linear velocity of the orifices is nearly $u = \sqrt{2gh}$ and that the efficiency is then $e = 0.66$.

(4) A 1″ nozzle delivers a cubic foot a second. What dynamic pressure will it produce against a plane normal to the jet, and what will the pressure be if the plane is inclined 30°, 45°, and finally 60°, supposing all the water to be guided in the direction of maximum deflection?

(5) A stream of 64 square inches section and with a velocity of 40 f. s. impinges on the vertex of a fixed cone of 45° semi-vertical angle and in the direction of its axis. What is the force acting on the cone? If a sphere replaces the cone, what is the action?

(6) Water flows at the rate of 2 cubic feet a second through a flexible hose of 3″ diameter. If the hose is lashed to two eye bolts so that the parts leading from them to the bend in the hose make an angle of 60°, the rest of the hose being free, what is the pull on each bolt, and what is the whole dynamic pressure on the hose between the bolts?

(7) Water is flowing at the rate of 8 f. s. through a 2″ pipe. The sudden closing of a valve stops the water in a 200-foot length of the pipe in 0.10 seconds. What is the increase of pressure near the valve?

(8) Two cubic feet of water moving 20 f. s. impinge on a stationary vane and are thereby deflected 60°. What is the pressure on the vane in the direction of and normal to the jet?

(9) If a jet impinges normally on a succession of plane surfaces which move away in the direction of the impulse, what must the relative velocity of the vanes be for maximum efficiency and what is that efficiency?

(10) If a series of curved vanes are successively impinged upon by a jet and move away in the direction of the impulse, how should they be shaped, and with what relative speed should they move to secure maximum efficiency, and what is that efficiency?

(11) A 2″ nozzle, under a head of 80′, and with a coefficient of velocity 0.94, delivers its jet normally against a series of small plane vanes moving in a circumference of 16′ diameter. How fast should the wheel be made to turn and what then is its power?

(12) A 1″ nozzle, under a 100′ head, coefficient of velocity 0.98, delivers its jet against a series of cups moving in a circle of 2′ diameter and so shaped that they reverse the direction of the impinging water. How many revolutions should the wheel make, and what would its horse-power then be?

(13) A "hurdy-gurdy" wheel of 6′ diameter is actuated by the jet from a 1″.89 nozzle under a head of 386′, the water being conveyed to the nozzle through 6900′ of 22″ pipe. If the efficiency of the wheel was 0.87, what was its H. P.?

(14) Compute the power and efficiency of a water wheel of the following characteristics: $r = 2'$; $r_1 = 3'$; $\alpha = 30°$; $\varphi = 60°$; $\beta = 120°$, when actuated by 2.2

cubic feet of water per second at 100 f. s., and when the speed of the wheel is 184 revolutions a minute.

(15) Given $r = 2'$; $r_1 = 3'$; $\alpha = 45°$; $\varphi = 90°$; and $\beta = 150°$; what is the work of a water wheel and what is its efficiency when actuated by 2.2 cubic feet of water per second at 100 f. s.; the speed of the wheel being 337.5 turns a minute?

CHAPTER XI.

Hydraulic Machines—Work of Expanding Gases.

(114) When the energy of a natural head of water is to be transformed into useful work, the water is collected in a reservoir, canal, or head race; is thence conveyed through a pipe, penstock, or flume to the hydraulic motor; and, after passing through or over the latter, issues into the lower level, or tail race.

The hydraulic motor generally consists of a wheel which is caused to revolve either by the descending weight of the water or by its impulse, and such a motor is called a "water-wheel" when the water enters it at only a part of its circumference, and a "turbine" when the water enters around its entire circumference.

(115) The "overshot" water-wheel is commonly used for falls of considerable height. Its motive force is principally the direct action of the weight of the water which is received in its buckets near the top of the fall and descends in them until emptied in the tail race. These wheels are sometimes 60′ and more in diameter, and have an efficiency from .70 to .90, one of their special advantages being that when the water supply runs low their efficiency is highest on account of the buckets being only partly filled and so the loss of water by spilling before the bottom of the fall is reached being a minimum.

The "undershot" water wheel is operated by the impulse of water which flows nearly horizontally beneath it and acts upon its radial vanes. These wheels

commonly have an efficiency from .20 to .40, and which is least when they are turned by the current of an unlimited stream, as in the case of a floating mill anchored in a river. When the vanes are curved so as to make the water leave with a less absolute velocity than that of the wheel itself, the undershot wheel may have an efficiency as high as 0.60.

"Breast" wheels operate partly by the weight and partly by the impulse of the water. In them the buckets of the overshot wheel are replaced by vanes which move in a masonry channel partly surrounding the wheel. There is, of course, considerable leakage past the vanes, which cannot fit the channel closely, and the efficiency of the breast wheel varies from .50 to .80. These wheels must be of greater diameter than the height of the fall, and so are only used for comparatively small falls.

Other water-wheels, which are actuated by the impulse of a jet directed against their vanes, are known as "impulse" wheels, "vertical," or "horizontal," according to the position of the wheel. Horizontal impulse wheels, further, are "outward flow," or "inward flow," wheels, according to whether the horizontal jet actuating them enters the wheel at its inner and leaves at its outer circumference, or vice versa. There are also horizontal "downward flow" wheels, actuated by one or more jets which pass down over the vanes without either approaching or receding from the wheel's axis. These impulse wheels have efficiencies from .75 to .85.

(116) Turbines, as already stated, are wheels in which the water enters around the entire circumference so that all the vanes are simultaneously acted

upon. They are usually horizontal, and may be "outward," "inward," or "downward" flow turbines, according to how the water passes through them. In all turbines the water passes through fixed curved guide passages which give it the proper direction to make it enter the wheel passages without shock, and the latter are also curved so as to cause the water leaving the turbine to have as little remaining energy as possible.

Turbines are also divided into the two classes of "impulse" and "reaction" turbines, though any turbine will act either as a reaction or as an impulse turbine according to whether the supply of water completely fills the wheel passages or not. If the wheel passages are only partly filled with water the action is entirely impulsive, but when the water fills them it transmits more or less static pressure into the wheel. The theory of impulse turbines is the same as that of impulse wheels, and the shape of their guide and wheel passages is determined as set forth in the last chapter. In reaction turbines, however, where the discharge of water, just as in the Barker's mill, varies with the speed of rotation and with the cross sectional area of the wheel passages, the problems of design and of the theoretical determination of efficiency are more difficult, and will not here be entered upon.

The efficiency of well designed turbines, as experimentally measured, is from .70 to .80.

(117) Every hydraulic motor may be made to raise water, or act as a pump, by the application of power to give it a reversed motion. The reversed overshot water-wheel, which dips up water in its buckets and

raises it to a height somewhat less than the diameter of the wheel is known as a "Chinese wheel." A reversed breast wheel is known as a "scoop" or "flash" wheel. The reversed inward flow turbine is the well-known "centrifugal" pump.

The different conditions, however, under which motors and pumps work sometimes makes considerable differences in design necessary for efficiency in the two cases.

(118) As a rule water pressure engines are operated by an artificial head produced by steam pumps working in connection with an accumulator, though sometimes they are used with natural heads of water. They are of various forms, single or double acting, and with one or more cylinders, a not uncommon form being three single acting oscillating cylinders, inclined at 120° to each other, and driving the same crank; but in all cases the principle is the same, and it may be fully illustrated by a consideration of the simple ram, such, for example, as is used to lift a weight by direct action to a height equal to the stroke.

Let p_0 be the pressure in the accumulator, p that in the working cylinder, A the piston area, and V the piston speed when lifting a weight W. Then the useful work per second is $WV = ApV$, and p is less than p_0 by an amount which measures the sum of all the losses of head between the accumulator and the piston. But the total loss of head can be expressed in the form $k\frac{V^2}{2g}$, and so we have $\frac{p_0 - p}{w} = k\frac{V^2}{2g}$, or $p = p_0 - kw\frac{V^2}{2g}$; whence the useful work per second is—

$$(46) \qquad E = WV = A\left(p_0V - kw\frac{V^3}{2g}\right).$$

If the weight being lifted is reduced, diminishing the useful work being done, V increases, and so too does the loss of head $k\frac{V^2}{2g}$; and, in the limiting condition $W = 0$, V reaches the maximum value $\sqrt{\frac{2gp_0}{wk}}$, and the loss of head is $\frac{p_0}{w}$ or the total head due to the accumulator pressure.

Thus a water pressure engine is automatically self-regulating as regards speed, the hydraulic resistance increasing as the load is diminished.

Again, solving equation (46) for V, and putting Ap for W, we see that for any given load there is a certain "speed of steady motion" given by the equation

$$(47) \quad V = \sqrt{\frac{2g\,(p_0 - p)}{kw}},$$

and, since k can be increased at will by more and more nearly closing the valve between the accumulator and the cylinder, the speed of a water pressure engine can be regulated at pleasure.

Furthermore, by differentiating (46) we find that the power is a maximum when $V = \sqrt{\frac{2gp_0}{3\,kw}}$, which is $\frac{1}{\sqrt{3}}$ times the speed under no load, so that the power of a water pressure engine supplied by a given pipe is greatest when one-third the total head is lost in overcoming frictional and hydraulic resistances.

(119) It should be noted that the piston must start from rest and so there must be accelerated motion during the first part of each stroke, which reduces the effective pressure in the cylinder. Moreover all

the water in the cylinder and supply pipes, as well as the piston and the weight it lifts, must be accelerated, and, since the pipes are smaller than the cylinder, the water in them must have a correspondingly greater acceleration than that of the piston, so that there must be a material reduction in the effective pressure while the inertia is being overcome, and so a considerable time may elapse before steady motion is attained.

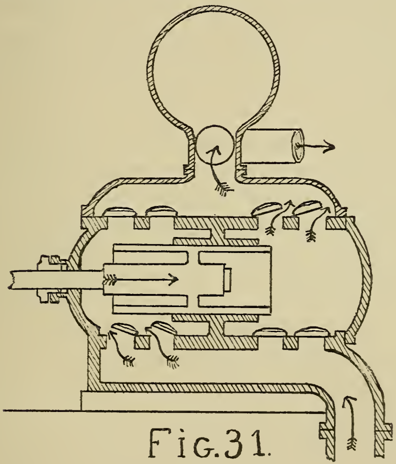


Fig. 31.

So, too, if the motion is a reciprocating one, there is a similar retardation and increase of effective pressure during the latter part of each stroke. Thus a water pressure engine is equivalent to a machine with very heavy moving parts, and is liable to considerable variation of pressure. This, together with the incompressibility of the water, renders it necessary to apply

relief valves or air chambers to such a system to prevent excessive pressures.

(120) Pumps of common form are in principle merely the reversal of water pressure engines, the main difference being that in the former the valves open and close automatically by the action of the water while in the latter the valves are either worked by hand or by a connection with some moving part of the machine.

Fig. 31 shows a double-acting force pump with an air chamber to maintain a continuous delivery, and which is actuated by a steam cylinder (not shown) whose piston rod is an extension of that of the pump itself.

(121) The hydraulic cylinder is often used as a brake, or means of bringing a moving weight to rest without shock. In its simplest form the hydraulic brake is a cylinder full of liquid fitted with a piston pierced with one or more holes, so that motion of one part with reference to the other forces the liquid through the holes from one side of the cylinder to the other, thereby producing a pressure which is evidently proportional to the square of the velocity of relative movement. Such a device by itself would never bring a moving weight to rest, and moreover has the disadvantage of causing a maximum pressure, when the velocity is greatest, some four or five times the uniform pressure which would produce the same result. It is therefore now usual to make the escape openings from one side of the piston to the other of varying cross section, so that the escape area is greatest at the beginning of the motion to be checked and is gradually diminished in such a way as to make the

retarding pressure in the cylinder uniform. This is readily accomplished by having the openings in the form of shallow grooves in the inner surface of the cylinder, either the depth or the width of the grooves being gradually reduced from one end of the cylinder to the other.

(122) Thus far the work done by liquids only has been considered, and something must be said in regard to the energy of gases.

It has been pointed out that a confined gas possesses a store of energy within itself and, unlike a liquid, performs useful work by expanding. We will now show that this internal energy is, for any given mass of any gas, a function of its temperature alone, and does not depend upon either its volume or its pressure.

As stated in Chapter I, experiment has shown that for all gases, not near the state of liquifaction, the product of the pressure and volume of any given mass divided by its absolute temperature is constant. When unit mass is taken and this relation expressed in the form

$$(48) \quad pv = RT.$$

it is called, the characteristic equation of the gas considered. Thus, since a cubic foot of air at 32° F. and 14.7 pounds per square inch pressure weighs .0807 pounds, $R = \frac{p_0 v_0}{T_0} = \frac{14.7 \times 144}{492 \times .0807} = 53.3$, and for air, $pv = 53.3\ T$, in which p is the pressure in pounds per square foot and v the volume in cubic feet of one pound of air at an absolute temperature of T° F. and, too, R for any other gas is evidently found by dividing 53.3 by the specific gravity of the gas, thus being for coal gas, the density of which is .43 that of air, 123.95.

(123) If now we suppose the pressure p to remain constant while the volume changes dv, (48) shows that the temperature will change $\frac{pdv}{R}$, and so the gas must have received the quantity of heat $\frac{c'pdv}{R}$ when c' is the specific heat of the gas at constant pressure.

Also, supposing the volume v to remain unchanged while the pressure varies dp, (48) shows a change of temperature $\frac{vdp}{R}$ and consequently the absorption of a quantity of heat $\frac{cvdp}{R}$, where c is the specific heat at constant volume.

Therefore, when both volume and pressure change, the quantity of heat received by the gas is

$$(49) \qquad dq = \frac{1}{R}(c'pdv + cvdp).$$

Now, differentiating (48), we get

$$(50) \qquad pdv + vdp = RdT$$

and, by eliminating dp from (49) and (50),

$$(51) \quad dq = cdT + \frac{c' - c}{R} pdv.$$

(124) Considering (51), the quantity pdv is the external work done by the gas in changing its volume by the amount dv, for the work done by each element of the surface enclosing the gas is the product of its area, the pressure, and the distance it moves normal to itself, and, the pressure being constant over the entire surface, the sum of the works done by all the elements of the surface is the product of the pressure and the change of volume. Now, integrating (51), and calling T_0 and T_1 the initial and final temperatures and

W the total external work done by the elastic force of the gas, we have

$$(52) \qquad q = c(T_1 - T_0) + \frac{c' - c}{R} W,$$

which gives the quantity of heat absorbed by unit mass of a gas when it changes temperature from T_0 to T_1 while doing the work W, and by which, if the foot, the pound, and the degree Fahr. are the units used, q will be given in British thermal units.

If in (52) we make $T_1 = T_0$, implying that the gas has the same temperature in its final as in its initial state, we get $W = \frac{R}{c' - c} q$, so that $\frac{R}{c' - c}$ is what is known as the mechanical equivalent of heat and usually designated by J. Thus in the case of air, taking $R = 53.3$ as already calculated, $c' = .2375$ as found by Regnault, and $\gamma = \frac{c'}{c} = 1.408$, we get $J = 775$, a near approximation to the value 779 which the best recent experiments assign to it. It should be observed, therefore, that, while the value of R is different for different gases, the theory of the conservation of energy requires the quantity $\frac{R}{c' - c}$ to be independent of the nature of the body which serves as intermediary in the conversion of heat into mechanical work or of work into heat.

If, then, no heat is either gained or lost by the gas, so that $q = o$, we have $W = cJ(T_0 - T_1)$, showing that the external work which a gas can do depends only upon its change of temperature, and has a maximum possible value cJT_0, which therefore is the internal energy of unit mass of gas at temperature T_0.

(125) In isothermal expansion, then, the work done by any gas exactly equals the heat imparted to it to maintain its temperature, and in adiabatic expansion the work done is the product of the change of temperature by the specific heat of constant volume, in both cases the work being in heat units and to be multiplied by 779 to get its value in foot-pounds.

Frequently, however, we desire to determine the work done by a gas in changing volume when the temperature is not part of the given data. If the change is isothermal we have then only to put $dT = o$ in (50) and by integrating we get $pv = k = p_1v_1$, whence the work done in expanding from v_1 to v_2 is $\int_{v_1}^{v^2} p\,dv = \int_{v_1}^{v_2} p_1v_1 \frac{dv}{v} = p_1v_1 \log_e \frac{v_2}{v_1} = p_1v_1 \log_e \frac{p_1}{p_2}$, so that the work done by any gas in changing volume isothermally is given by

$$(54) \qquad W = p_1v_1 \log_e \frac{v_2}{v_1} = p_1v_1 \log_e \frac{p_1}{p_2}.$$

If, on the other hand, the change is adiabatic, we get, by putting $dq = o$ in (49), $\frac{dp}{p} + \frac{c'}{c} \cdot \frac{dv}{v} = o$, or by integration, $\log p + \frac{c'}{c} \log v = k_1$, $pv^\gamma = k_2 = p_1v_1{}^\gamma$. Whence the work done in expanding from v_1 to v_2 is $\int_{v_1}^{v_2} p dv = p_1v_1{}^\gamma \int_{v_1}^{v_2} \frac{dv}{v^\gamma} = \frac{p_1v_1{}^\gamma}{\gamma - 1}\left[\frac{1}{v_1{}^{\gamma-1}} - \frac{1}{v_2{}^{\gamma-1}}\right] = \frac{p_1v_1 - p_2v_2}{\gamma - 1}$, so that the work done by any gas in changing volume adiabatically is given by

$$(55) \qquad W = \frac{p_1v_1 - p_2v_2}{\gamma - 1} = \frac{p_1v_1}{\gamma - 1}\left[1 - \left(\frac{v_1}{v_2}\right)^{\gamma-1}\right] = \frac{p_1v_1}{\gamma - 1}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right].$$

Also, since $\frac{p_1 v_1}{T_1} = R = \frac{p_2 v_2}{T_2}$, $\frac{p_1 v_1}{p_2 v_2} = \frac{T_1}{T_2}$, or, since $p_1 v_1^\gamma = p_2 v_2^\gamma$, $v_2 = v_1 \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma - 1}}$ and, similarly, $p_2 = p_1 \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma - 1}}$, in all of which γ is the ratio $\frac{c'}{c} = 1.408$, and, if p is put in pounds per square foot and v in cubic feet, W will be the work in foot-pounds.

(126) It must be noted that expressions (54) and (55) for work done in isothermal and in adiabatic expansions are the total work done in changing volume, and that to find the available work it is necessary to subtract whatever work is done in overcoming the resistance of the back pressure. Moreover, in case the gas enters the working cylinder at a constant pressure, it does work by transmitted pressure, just as a liquid would, and this must be added to the work of expansion.

Similarly, when work is done upon a gas, compressing it, the helping action of the back pressure must be taken account of, and so, too, must the work necessary to be done, after complete compression, in sweeping the compressed gas out of the cylinder.

Thus suppose a volume v_1 of gas at constant pressure p_1 to enter a cylinder, and the supply to be then cut off and the gas allowed to expand adiabatically to volume v_2 and pressure p_2, after which the exhaust is opened, the volume v_1 admitted on the other side of the piston, and the process reversed, the result being a reciprocating motion which, by proper mechanical connections, performs useful work. Then at each stroke the gas does the work $v_1 p_1$ in entering, and the work $\frac{v_1 p_1 - v_2 p_2}{\gamma - 1}$ in expanding, but during the whole

stroke the back pressure p_2, due to the gas being expelled from the other side of the cylinder, opposes the motion, and so the work v_2p_2 is expended, leaving as the total available work $v_1p_1 + \frac{v_1p_1 - v_2p_2}{\gamma - 1} - v_2p_2 =$ $\frac{\gamma}{\gamma - 1}(v_1p_1 - v_2p_2)$.

Similarly, supposing the work done to consist in compressing a volume v_1 of gas to v_2 and then forcing it into a reservoir, then the work of compression is $\frac{v_2p_2 - v_1p_1}{\gamma - 1}$, and the work of pushing the compressed gas out of the cylinder is v_2p_2, but the back pressure p_1 helps to do all this, and so reduces the expenditure of energy by v_1p_1, making the total work done $\frac{\gamma}{\gamma - 1}(v_2p_2 - v_1p_1)$.

If, however, the gas enters the working cylinder direct from a reservoir in which the pressure is not maintained, so that expansion takes place from the instant that admission of gas to the cylinder begins, then there is no work of entrance, but only the work of expansion given by (54) or (55) as the case may be, and in either case requiring to be reduced by the work done in overcoming the back pressure.

Evidently it is unnecessary to consider each stroke of the piston separately, where the same cycle of operations is being repeated, and we can determine the whole available energy of a reservoir of compressed gas by applying our formula, taking v_1 as the whole volume of the reservoir.

(127) As an example, we will determine the available energy per pound of air at 62° F. in a flask at

1500 pounds pressure per square inch. Here $T_1 = 522°$; $p_1 = 1500 \times 144 = 216{,}000$; and so $v_1 = \frac{53.3\ T_1}{p_1} = .1288$ cu. ft. The air will give up all its available energy when it is expanded to atmospheric pressure, so we have $p_2 = 14.7 \times 144 = 2116$, and the work done in the expansion is $\frac{p_1v_1 - p_2v_2}{\gamma - 1}$, which, since $v_2 = v_1 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}} = 3.44$, equals 50,345 foot-pounds. But of this work a portion, $p_2\ (v_2 - v_1) = 2116 \times 3.3112 = 7007$ ft. lbs., is used in overcoming the atmospheric pressure during expansion, and so the total available energy is $50{,}345 - 7007 = 43{,}338$ ft. lbs.

(128) As another example, we will take the work per stroke of a steam engine with piston area A, stroke a, and cut off at $\frac{1}{n}$ the stroke, and we will assume the expansion to be isothermal, which is not far from the truth when wet steam is used. Then, if the entering pressure is m atmospheres, the work in entrance is $\frac{aA}{n} mp_0$, and the work of expanding to the end of the stroke is $\frac{aAmp_0}{n} \log_e n$. But the work used in overcoming the back pressure p_0 is aAp_0. Therefore the useful work per stroke is $aAp_0 \left(\frac{m - n}{n} + \frac{m}{n} \log_e n\right)$, p_0 being the atmospheric pressure.

Problems XI.

(1) The density of hydrogen is $\frac{1}{14}$ that of air; what is the characteristic equation for hydrogen, and what

would be the pressure of 5 pounds of it contained in 3 cubic feet and at 50° F.?

(2) What is the weight of the air in a room 20′ by 18′ by 10′, the temperature being 62° F. and the pressure 14.7 pounds per square inch?

(3) The specific gravity of steam, compared with air, is .622. What is the characteristic equation for steam, and what volume of steam results from vaporizing 1 cubic inch of water at 212° F.?

(4) A hydraulic ram 6″ in diameter is supplied by a 2″ pipe from an accumulator in which pressure is kept at 600 pounds per square inch. It takes 10% of the accumulator pressure to overcome friction of the mechanical parts and the total hydraulic resistance is 10 times the velocity head in the supply pipe when a force of 12,000 pounds is being exerted by the ram. What is the speed of steady motion? If the speed is doubled, what force is being exerted by the ram?

(5) If in the preceding example the partial closing of the supply valve doubles the hydraulic resistance, what then is the speed of steady motion?

(6) A weight of 10 tons is moving 5 f. s. and it is desired to reduce its speed to 1 f. s. in a travel of 1′ by means of a hydraulic brake having a piston of 16 square inches effective area. What must the escape area from one side to the other of the piston be?

(7) If a hydraulic brake is to oppose constant resistance to motion, so as to uniformly retard a weight W which starts with velocity V and to bring it to rest in the distance l, deduce a formula for the escape area a, the piston area being A.

(8) Air at 62° F. and 15 pounds pressure per square inch is suddenly compressed to one-fifth its volume;

what do its temperature and pressure become? What will the pressure be when it is cooled again to 62° F.? What will its temperature be if, after cooling, it is allowed to expand adiabatically to its original pressure?

(9) Compare the work done in compressing each pound of air as in example (8) with the work it will do, after cooling, in again expanding to atmospheric pressure.

(10) Compare the work of obtaining a pound of air at 4 atmospheres pressure and atmospheric temperature (62° F.) by compression adiabatically and isothermally.

(11) The cylinder of a compressed air engine is of 1 square foot cross section and the stroke is 18″. If the initial pressure is 3 atmospheres and the temperature 62° F., what should be the point of cut-off so that the pressure may be atmospheric at the end of the stroke? What then is the available work done per stroke, and also per pound of air, and what is the temperature of the exhaust air?

(12) Compare the work of obtaining a pound of air at 100 atmospheres pressure and at atmospheric pressure by one compression, and by three successive equal compressions with time for cooling the air between them.

(13) The air flask of a 5-meter Whitehead torpedo contains 11.77 cubic feet of air at 1500 pounds per square inch. What is its available energy?

(14) If the air of example (13) is used at a constant pressure of 300 pounds per square inch by means of a reducing valve, what is the available energy?

(15) If in example (14) the expansion in the working cylinders of the engine is incomplete, the cut-off

being at one-third the stroke, what is the available energy?

(16) A pound of gunpowder in burning forms $\frac{4}{10}$ pounds of gas and gives up 1400 B. T. units. Supposing the solid and liquid residue to give all their heat to the gun walls, so that the gas expands adiabatically, how many foot-tons of energy will a pound of powder supply in expanding three times?

(17) If the combustion of a pound of smokeless powder converts it all into gas and develops 2000 B. T. units, how many foot-tons of work would be done per pound of powder in expanding four times, and with that expansion in a 6″ gun, what velocity would 25 pounds of such powder give to a 100-pound projectile?

(18) Prove that in supplying a reservoir with compressed air by machinery which does the compression in n stages, with complete cooling after each stage, it is most economical to do the same amount of work at each compression, and, if k is the ratio of initial to final volume, to reduce the volume in the ratio $\sqrt[n]{k}$ at each compression.

ANSWERS TO PROBLEMS.

Problems I.—(1) 288 pounds. (2) 3.86 pounds. (3) 491.07 pounds. (4) 736.6 pounds. (5) + 4 pounds. (6) 900 pounds. (7) 50 pounds. (8) 2592 pounds. (9) 286.4 pounds per sq. in. (10) 10,056 tons. (11) 1678 pounds per sq. in. (12) 110.2 cu. in., 16.53 pounds per sq. in. (13) 2369 cu. yds. (14) 40.12 pounds per sq. in.; $v_1 = 0.497v_0$. (15) 300 and 1018.5 pounds per sq. in. (16) Spherical. (17) 91.3 pounds, 1201 cu. ft. (18) About 400,000 ft. lbs.

Problems II.—(1) 32.36 pounds; 4660 pounds. (2) 33.′87; 33.′08. (3) 2666.7 pounds per square inch. (5) 360 pounds. (6) 5625 pounds. (7) 0.542 pounds per sq. in. (8) 0.276 pounds per sq. in. (10) $2 + \sqrt{3}$. (11) $106\frac{2}{3}$ ft. (12) $v_1 = 2.201v_0$. (13) $v_1 = \frac{v_0}{1 + .00241\,x}$, where x is depth in inches of water surface in cone. (14) $v_1 = \left(\frac{v_0}{1 + .00241\,x}\right)^{\frac{1}{\gamma}}$. (16) 6934′. (17) 23876′. (18) 27613′. (19) 32.′28. (20) (a) It will still just float (neglecting a very small increase of weight of envelope due to decreased density of air it displaces). (b) It would sink. (21) (h — 414.72)A cu. in.

Problems III.—(1) 2:1. (2) 62,071 pounds. (3) 1:2. (4) At its mid height. (5) So as to cut off $\frac{3}{4}$ the opposite side. (6) 6″.13 in the water. (7) Upper faces equal and each half that on each lower face. (8) 50,000 pounds. (9) 19′.76. (11) 1:15.6. (12) 7′.

(14) The lower corner supports $\frac{1}{2}$ and the others each $\frac{1}{4}$. (15) $\frac{4h}{7}$. (16) $346\frac{2}{3}$ pounds. (17) 13.′44 and 10.′38. (18) 9.′43 and 7.′26. (19) 1″ below its center. (20) $\frac{3}{8}a$ below surface. (21) $\frac{7ab}{6\sqrt{a^2+b^2}}$ below surface. (22) $\frac{a^2}{4h}$, where h is depth of center. (24) $h = \frac{9}{2}h_0$, where h_0 is barometric height. (25) Half the height of the cone.

Problems IV.—(1) 0.4903 pounds. (2) 3.144 cu. ft. (3) 5.926 cu. ft. (4) 417.5 and 370 pounds. (5) 11.44. (6) 21.505 and 0.914. (7) 0.771. (8) 147 pounds. (9) 19.5. (10) 18.41. (11) 1.0952, 0.9747; 1.0000. (12) 0.87 copper, 0.13 tin. (13) 20. (14) 0.2516 cu. ft. (15) $a(1 - \frac{2}{3}\sqrt[3]{3})$. (16) .03118 cu. ft. or 53.87 cu. in. (17) 3.5 grains. (18) 2.5955 silver to 1 copper; or 11.55 to 4.45. (19) 11 : 9 weight, 2 : 3 vol. (20) wrh^2 acting $\frac{2h}{3}$ below top. (21) $wr^2h\left(2+\frac{\pi}{2}\right)$ acting through center of axis. (22) $\frac{wrh}{6}\sqrt{9h^2+\pi^2r^2}$ making $\tan^{-1}\frac{3h}{\pi r}$ with axis. (24) $\frac{\sqrt{13}}{3}w\pi r^3$ through center. (25) $\frac{w\pi r^3}{3}\sqrt{13-12\cos\theta}$ and $\frac{w\pi r^3}{3}\sqrt{13+12\cos\theta}$. (26) 443 f. s.; 13.755 pounds per sq. in.; 1830′; 3384′; inside same as before; outside, 12.95 pounds per sq. in. (27) $\frac{w\pi r^2}{3}\left(h - \frac{15r\sin 2\alpha}{8}\right)$ and $\frac{5w\pi r^3\cos^2\alpha}{4}$. (28) $\frac{8w\pi r^3}{3\sqrt{10}}$. (29) $\frac{2wr^3}{3}$.

Problems V.—(1) 2.13 f. s.; 5496′; 1680′; 25 pounds. (2) 135.86 pounds; 2298′; 9180′. (3) Draft is $h\left(1-\sqrt[3]{1-\frac{\rho_1}{\rho}}\right)$. (4) $\frac{5W}{2}$. (5) 2388 pounds. (8) $h\left(1-\sqrt{1-\frac{1}{n}}\right)$. (9) Just immersed. (10) 4267 tons. (11) 24′ — 2″.46; 293 tons. (15) $h=\frac{a\sqrt{2}}{2}$. (19) $\rho > \sec^4 \alpha$, when α is semi-vertical angle. (21) $GM=\frac{a}{4}$.

Problems VI.—(1) $\frac{\pi a^4\omega^2}{4g}$. (2) $\omega^2=\frac{gh}{b^2}$. (3) $\omega^2 = \frac{4g\left(\frac{2\pi a^3}{3}-V\right)}{\pi a^4}$. (4) $a-\frac{2g}{\omega^2}$ vertically. (5) $\frac{\omega^2}{2g}(\overline{OA}^2 - \overline{OB}^2)$. (6) $\frac{\pi r^4\omega^2}{4g}$ w; $\pi r^2\left(\frac{\omega^2 r^2}{4g}+h\right)$ w; and $\pi rh\left(\frac{\omega^2 r^2}{g}+h\right)$w. (7) It will be emptied by rotation $\omega^2=\frac{gh}{ar}$ where a is distance from axis of cone to axis of rotation. (8) 23′.04; 5′.06; 28′.10. (9) 12.8 f. s. and 7.11 f. s.; 2′.56 and 0′.79; 3′.44 and 4′.21; 2.60 and 2.17 pounds per sq. in.; 1.49 and 1.83 pounds per sq. in. (10) 0.98 pounds per sq. in. below atmospheric pressure. (11) 66.51 f. s.; 4.24 pounds per sq. in. (12) 5.66 f. s.; 17.89 f. s.; 80 f. s. (13) 46.99 f. s.; 49.96 f. s.; 92.61 f. s. (14) 121.56 f. s.; 122.75 f. s.; 145.42 f. s. (15) 56.57 f. s. (16) 100.5 f. s. (17) 121.96 f. s. (18) 42.64 f. s. (19) 14.73 f. s. (25) $\omega^2=\frac{g}{a}$.

Problems VII.—(1) 363.7 cu. ft. (2) 1′.232. (3) 366 cu. ft. (4) $\frac{4cA}{5}\sqrt{2ga}$. (5) $\frac{8cA}{15}\sqrt{2ga}$. (6) $\frac{3\pi cA}{16}\sqrt{2ga}$.